EFFICIENT ESTIMATION SCHEME FOR SPIN-AXIS ATTITUDE DRIFT IN THE KEEPING PHASE

K Kaku & M Horii

Tsukuba Space Center
National Space Development Agency of Japan
Ibaraki, Japan

ABSTRACT

This paper presents an efficient estimation scheme for the attitude determination of spin-stabilized geostationary satellites in the keeping phase. We derive an estimation scheme based on an attitude drift model. In this model, parameters (i.e., attitudes of epochs, attitude drift rate and direction, and biases of measurements) are estimated by the least square method from the measurements of nadir angles. Furthermore, the results of the verification of this scheme by the telemetry data of the satellite on orbit is explained. Also a future application plan of this scheme will be explained.

Keywords: Solar Radiation Pressure, Attitude Drift, Drift Rate, Drift Direction, Attitude Jump, Least Square Method

1. INTRODUCTION

An attitude of geostationary satellite suffers a torque due to solar radiation pressure, which is caused by the electromagnetic waves emitted by the Sun from X-ray to radio waves. This torque causes an attitude drift to a spin-stabilized geostationary satellite (Ref. 1).

On the other hand, the attitude of a geostationary satellite in the keeping phase has to be kept in a required region in order to achieve the mission requirement. Then, we must estimate the attitude drift and control the attitude with a certain period. The accuracy with which the spin-axis attitude can be determined is one of the key elements that specify the period of the attitude control manoeuvres.

Provided that the required region of attitude keeping is sufficiently wide, attitude determination may be done without considering the attitude drift, i.e., using the attitude determined by the constant attitude model in which the attitude is constant with respect to time. This is the reason why few discussions have been made so far about the attitude determination in the keeping phase in contrast with many discussions about that in the transfer orbit phase (Refs. 2-4). In fact, to date, all the attitudes of the spin-stabilized geostationary satellites launched by National Space Development Agency of Japan (NASDA) were successfully kept in a required region by means of the constant attitude model. For example, the attitude of Geostationary Meteorological Satellite-3 (GMS-3), which was launched in 1984 and is now in operation, is kept within the tolerance of 0.5 degree from the anti-orbit-normal direction, and that of Communications Satellite-2 (CS-2a and CS-2b) launched in 1983 were kept within 0.08 degree in the initial phase.

Provided that the required region of attitude keeping becomes narrower, however, the attitude drift has to be considered in order to improve the attitude determination accuracy. The benefits of considering the attitude drift are not only to make the interval of attitude controls longer, but also to make a precise attitude control possible, which has been difficult by the constant attitude model.

2. ALGORITHM

2.1 Attitude drift model

2.1.1 Basic equations. In modeling the attitude drift, the following assumptions are made:

1) Geostationary spin-stabilized satellites in the keeping phase are considered. Then, the attitude is assumed to be almost perpendicular to both the equatorial plane and the orbital plane.
2) Spin-axis attitude equals to the angular momentum vector of a satellite. Torque due to solar radiation pressure, the only force that we concern, acts on a point on the spin-axis of the satellite.
3) Spin-axis attitude drifts with a constant drift rate along a fixed drift direction measured from the Sun direction over the time interval concerned.
4) In order to treat the estimation time span in which attitude control manoeuvres are contained, we set attitude jumps, i.e., the instantaneous attitude changes which correspond to attitude control manoeuvres. Here, it is

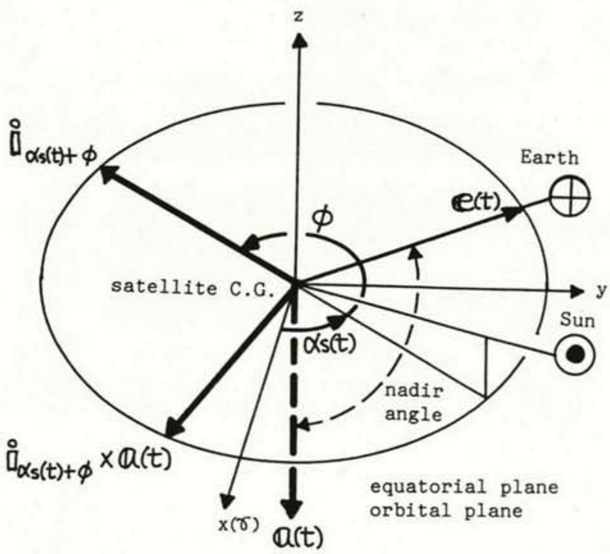


Figure 1. Illustration of the system

assumed that the attitude drift parameters do not change through the whole time span.

Let $[t_S, t_e]$ be the estimation time span. We divide $[t_S, t_e]$ as follows. Let $I_i = [t_{Ji}, t_{Ji+1})$ $(i=0,1,\dots,n)$, where $t_{J0} = t_S$, $t_{Jn+1} = t_e$. Here, t_{Ji} $(i=1,2,\dots,n)$ represents the attitude jump time. Then, the attitude drift equations may be written as

$$\frac{d\mathbf{a}_i(t)}{dt} = \omega\, \mathring{\mathbf{l}}_{\alpha_S(t)+\phi} \times \mathbf{a}_i(t) \qquad (1)$$
$$t \in I_i, \quad i = 0,1,\dots,n$$

where $\mathbf{a}_i(t)$ is a unit attitude vector in the time span I_i, ω is a constant that shows attitude drift rate, $\mathring{\mathbf{l}}_{\alpha_S(t)+\phi}$ is a unit vector defined by the Sun right ascension $\alpha_S(t)$ with a constant bias ϕ:

$$\mathring{\mathbf{l}}_{\alpha_S(t)+\phi} \equiv \begin{bmatrix} \cos(\alpha_S(t)+\phi) \\ \sin(\alpha_S(t)+\phi) \\ 0 \end{bmatrix} \qquad (2)$$

and $\times$ means a vector product. The attitude drift is specified thoroughly by the attitude drift rate ω and the attituide drift direction ϕ measured from the Sun direction through the whole span as shown in Figure 1.

2.1.2 Attitude propagation. By integrating Eq.1 with an integration step ΔT, the following solutions are obtained. Here, we assume that the Sun right ascension $\alpha_S(t)$ is a constant $\alpha_{S,t}$ in the interval $[t, t+\Delta T]$.

$$\mathbf{a}_i(t+\Delta T) \cong \mathbf{a}_i(t) + \omega\Delta T\, \mathring{\mathbf{l}}_{\alpha_S(t)+\phi} \times \mathbf{a}_i(t)$$
$$= A(t+\Delta T; t)\, \mathbf{a}_i(t) \qquad (3)$$
$$[t, t+\Delta T) \subset I_i, \quad i = 0,1,\dots,n$$

where

$$A(t+\Delta T;t) \equiv \begin{bmatrix} 1 & 0 & \omega\Delta T \sin(\alpha_{S,t}+\phi) \\ 0 & 1 & -\omega\Delta T \cos(\alpha_{S,t}+\phi) \\ -\omega\Delta T \sin(\alpha_{S,t}+\phi) & \omega\Delta T \cos(\alpha_{S,t}+\phi) & 1 \end{bmatrix} \qquad (4)$$

Thus, the attitude propagation from $\mathbf{a}_i(t)$ to $\mathbf{a}_i(t+\Delta T)$ is represented by the matrix $A(t+\Delta T;t)$. The overall attitude propagation matrix is defined as

$$P = P(t, t_{oi}, \omega, \phi) \equiv \frac{\partial \mathbf{a}_i(t)}{\partial \mathbf{a}_i(t_{oi})} \in SO(3) \qquad (5)$$
$$t \in I_i, \quad i = 0,1,\dots,n$$

where the time t_{oi} is the epoch for the interval I_i, and $\mathbf{a}_i(t_{oi})$ is the initial attitude of the attitude $\mathbf{a}_i(t)$. Then, P is obtained as follows. Let

$$B(t;t_{oi}) = A(t; t_{oi}+\ell\Delta T)A(t_{oi}+\ell\Delta T; t_{oi}+(\ell-1)\Delta T)$$
$$\dots A(t_{oi}+2\Delta T; t_{oi}+\Delta T)A(t_{oi}+\Delta T; t_{oi}) \qquad (6)$$

where

$$|t - (t_{oi}+\ell\Delta T)| \leqq \Delta T \qquad (7)$$

If $t < t_{oi}$ in Eq.6, ΔT must be replaced by $-\Delta T$ in Eqs.4,6,7. Matrix $B(t;t_{oi})$ defined by Eq.6 is not orthogonal, however, because of the calculation scheme. Then, we transform $B(t;t_{oi})$ to an orthogonal matrix by the Schmidt's orthogonalization method.

$$P = P(t, t_{oi}, \omega, \phi) = \text{orthogonalized } B(t;t_{oi}) \qquad (8)$$

Then, the attitude at an arbitrary time $t \in I_i$ is given by

$$\mathbf{a}_i(t) = P(t, t_{oi}, \omega, \phi)\, \mathbf{a}_i(t_{oi}) \qquad (9)$$
$$t \in I_i, \quad i = 0,1,\dots,n$$

2.2 Estimation model

The attitude drift is specified thoroughly by the parameters ω and ϕ. It should be recognized, however, that this representation has a singularity at $\omega=0$, i.e., if $\omega=0$, then ϕ can not be specified. Thus, we introduce the following variables in the process of computation to avoid this singularity.

$$\begin{cases} \xi \equiv \omega \cos\phi \\ \eta \equiv \omega \sin\phi \end{cases} \tag{10}$$

If the attitude jumps are set at time $t_{J1}, t_{J2}, \cdots, t_{Jn}$ in the estimation time span $[t_s, t_e]$, the parameters to be estimated are represented by the vector:

$$\mathbf{x} \equiv ((\mathbf{a}_0(t_{00}))^t, (\mathbf{a}_1(t_{01}))^t, \ \ldots \ , (\mathbf{a}_n(t_{0n}))^t, \ \xi, \eta, b_1, \dot{b}_1, b_2, \dot{b}_2)^t \tag{11}$$

which consists of the initial attitude $\mathbf{a}_i(t_{0i})$ in the interval I_i, the attitude drift parameters ξ, η, and the data biases b_1, $\dot{b}_1$, b_2, $\dot{b}_2$. Here, the initial attitudes $\mathbf{a}_i(t_{0i})$ (i=0,1, ... ,n) are always estimated and other six parameters can be estimated or be fixed separately.

Estimation is carried out by the least square method as follows. Let the observed data be $\{y_j(t_k);\ j=1,2;\ k=1,2,\ \ldots\ ,N\}$, then the parameters are estimated from the criterion

$$J \equiv \sum_{j=1}^{2}\sum_{k=1}^{N}\{y_j(t_k) - (\mathbf{a}(t_k)\cdot\mathbf{e}(t_k) - b_j - \dot{b}_j(t_k - t_{0i}))\}^2 \longrightarrow \text{minimum} \tag{12}$$

with

$$\mathbf{a}(t) \equiv \mathbf{a}_i(t) \quad \text{for } t \in I_i, \ \ i = 0,1,\ \ldots\ ,n \tag{13}$$

where the subscript j denotes the j-th attitude sensor and $\mathbf{e}(t_k)$ is the unit nadir vector at time t_k. The estimated parameters $\hat{\mathbf{x}}$ is given by

$$\hat{\mathbf{x}} = \lim_{p\to\infty} \hat{\mathbf{x}}^p$$

$$\hat{\mathbf{x}}^{p+1} = \hat{\mathbf{x}}^p + Q\sum_{j=1}^{2}\sum_{k=1}^{N}\mathbf{C}_{jk}^t\{y_j(t_k) - (\hat{\mathbf{a}}^p(t_k)\cdot\mathbf{e}(t_k) - \hat{b}_j^p - \hat{\dot{b}}_j^p(t_k - t_{0i}))\} \tag{14}$$

with

$$Q = \left(\sum_{j=1}^{2}\sum_{k=1}^{N}\mathbf{C}_{jk}^t\mathbf{C}_{jk}\right)^{-1}\Bigg|_{\mathbf{x}=\hat{\mathbf{x}}^p}$$

$$\mathbf{C}_{jk} \equiv \frac{\partial y_j(t_k)}{\partial \mathbf{x}}, \ j = 1,2, \ k = 1,2,\ \ldots\ ,N \tag{15}$$

where the subscript p denotes the value of the estimation parameters of the p-th iteration in the estimation process. Here, some calculation techniques are used. Since $\sum\mathbf{C}_{jk}^t\mathbf{C}_{jk}$ is singular, Q has to be calculated as

$$\begin{aligned} Q = \Big\{&\sum_{j=1}^{2}\sum_{k=1}^{N}\mathbf{C}_{jk}^t\mathbf{C}_{jk} \\ &+\varepsilon\sum_{i=0}^{n}(0,\ldots,(\mathbf{a}_i(t_{0i}))^t,\ldots,0)^t(0,\ldots,(\mathbf{a}_i(t_{0i}))^t,\ldots,0)\Big\}^{-1} \\ &-\varepsilon^{-1}\sum_{i=0}^{n}(0,\ldots,(\mathbf{a}_i(t_{0i}))^t,\ldots,0)^t(0,\ldots,(\mathbf{a}_i(t_{0i}))^t,\ldots,0) \end{aligned} \tag{16}$$

And, when some parameters should be fixed, the corresponding components have to be dropped from $\hat{\mathbf{x}}^p$, $\mathbf{C}_{jk}$, and $\sum\mathbf{C}_{jk}^t\mathbf{C}_{jk}$.

2.3 Observation model

Nadir angles are measured by two Earth horizon sensors mounted on the satellite. Then, the observation equations are given by

$$z_j = \cos^{-1}(\mathbf{a}\cdot\mathbf{e}) + b_j + \dot{b}_j(t - t_{0i}), \ j = 1,2 \tag{17}$$

with

$$\mathbf{a} = \mathbf{a}_i(t) = P(t, t_{0i}, \omega, \phi)\,\mathbf{a}_i(t_{0i}) \quad \text{for } t \in I_i, \ \ i = 0,1,\ \ldots\ ,n \tag{18}$$

where z_j is the nadir angle measured by the j-th attitude sensor, b_j is the initial value of the nadir angle bias, $\dot{b}_j$ is the change rate of b_j with respect to time, and $\mathbf{e}$ is the unit nadir vector at time t.

For easiness of computation we transform Eq.17 as follows. In the keeping phase it is reasonably assumed that the nadir angle is approximately 90 degrees and the nadir angle bias $b_j + \dot{b}_j(t - t_{0i})$ is sufficiently small, i.e.,

$$\begin{aligned} &\cos^{-1}(\mathbf{a}\cdot\mathbf{e}) \cong 90^\circ \\ &b_j + \dot{b}_j(t - t_{0i}) \cong 0^\circ, \quad j = 1,2 \end{aligned} \tag{19}$$

Using Eq.19, the following transformed observation equations can be obtained.

$$\begin{aligned} &y_j \equiv \cos z_j \\ &y_j = \mathbf{a}\cdot\mathbf{e} - b_j - \dot{b}_j(t - t_{0i}), \quad j = 1,2 \end{aligned} \tag{20}$$

where $\mathbf{a}$ is defined by Eq.13. In the estimation process of the previous section, y_1 and y_2 are regarded as the measurements.

For further calculations, the first-order derivatives of the observation equations $\mathbf{C}_j \equiv \partial y_j/\partial\mathbf{x}$ are necessary. Let the time t of data y_j be in $I_i = [t_{Ji}, t_{Ji+1})$, then they are calculated as

$$\mathbf{C}_j \equiv \frac{\partial y_j}{\partial \mathbf{x}} = \left(0,\cdots,0,\frac{\partial y_j}{\partial \mathbf{a}_i(t_{0i})},0,\cdots,0,\frac{\partial y_j}{\partial \xi},\frac{\partial y_j}{\partial \eta},\right.$$

$$\frac{\partial y_j}{\partial b_1}, \frac{\partial y_j}{\partial \dot{b}_1}, \frac{\partial y_j}{\partial b_2}, \frac{\partial y_j}{\partial \dot{b}_2}) \quad , \quad j = 1,2 \qquad (21)$$

The components of $\mathbb{C}_j$ are

$$\frac{\partial y_j}{\partial a_i(t_{oi})} = \frac{\partial}{\partial a_i(t_{oi})}\{P(t,t_{oi},\xi,\eta)\, a_i(t_{oi})\cdot e\} \sim e^t P^t(t,t_{oi},\xi,\eta)$$

$$\frac{\partial y_j}{\partial(\xi,\eta)} = e^t \frac{\partial}{\partial(\xi,\eta)} P(t,t_{oi},\xi,\eta)\, a_i(t_{oi}) \qquad (22)$$

$$\frac{\partial y_j}{\partial b_\ell} = -\delta_{j\ell}$$

$$\frac{\partial y_j}{\partial \dot{b}_\ell} = -(t - t_{oi})\delta_{j\ell} \quad , \quad \ell = 1,2$$

where $\delta_{j\ell}$ is the Kronecker's delta. The matrices $\partial P/\partial(\xi,\eta)$ are numerically calculated as

$$\frac{\partial P}{\partial \xi} = (\Delta\xi)^{-1}\{P(t,t_{oi},\xi+\Delta\xi,\eta) - P(t,t_{oi},\xi,\eta)\}$$

$$\frac{\partial P}{\partial \eta} = (\Delta\eta)^{-1}\{P(t,t_{oi},\xi,\eta+\Delta\eta) - P(t,t_{oi},\xi,\eta)\} \qquad (23)$$

For the interpretation of the derivatives $\partial y_j/\partial a_i(t_{oi})$, see Appendix.

3. VERIFICATION OF THE SCHEME

We carried out the verification of the scheme using the telemetry data of Communications Satellite-2b (CS-2b) which was launched in 1983. The results of the verification are summarized as follows.

3.1 Data span

Data span necessary for the estimation is examined. From the standpoint of the estimation theory the longer the span, the better. Provided that we use the long data span, however, there is a possibility that our assumption of constant drift parameters is no longer satisfied. The results are shown in Table 1. Judging from the converged value, it may be concluded that data span necessary for the estimation is at least three days and appropriately five days.

3.2 Accuracy

For the purpose of examining the estimation accuracy, we set a dummy attitude jump at time t_J in the estimation span $[t_s, t_e]$. Let $a_0(t)$ and $a_1(t)$ be the attitude in the time span $[t_s, t_J)$ and $[t_J, t_e)$, respectively. Then, the attitude jump $\Delta\theta(t_J)$ is defined by

$$\Delta\theta(t_J) = |a_1(t_J) - a_0(t_J)| \qquad (24)$$

Table 1. Estimated attitude drift parameters (CS-2b, 1983)

Used data span	Drift rate ω (degree/day)	Drift direction ϕ (degree)	Drift rate ω (degree/day) for fixed ϕ =0
One day			
Aug. 27	0.0699	349.92	0.0589
28	0.0386	333.54	0.0216
29	0.0271	354.49	0.0250
30	0.0265	61.63	0.0302
31	0.0313	32.76	0.0391
Two days			
Aug. 27,28	0.0298	34.41	0.0192
28,29	0.0182	340.11	0.0163
29,30	0.0142	9.90	0.0143
30,31	0.0173	4.17	0.0174
Three days			
Aug. 27-29	0.0168	9.41	0.0168
28-30	0.0145	348.53	0.0141
29-31	0.0141	359.35	0.0141
Four days			
Aug. 27-30	0.0151	3.77	0.0152
28-31	0.0142	349.56	0.0138
Five days			
Aug. 27-31	0.0141	359.68	0.0144

In principle $\Delta\theta(t_J)$ is 0, since the attitude jump at time t_J is dummy. If $\Delta\theta(t_J)$ is not 0, it may be attributed to the error of this scheme. Consequently, we obtained $\Delta\theta(t_J) < 0.01$ degree. Then, it should be concluded that the attitude determination accuracy of this scheme is about 0.01 degree.

3.3 Comparison with other results

Estimation result is showen in Figure 2 with two other results obtained by different methods. Estimated attitude by our model, which is marked by 'O' in Figure 2, has an uncertainty of ±0.01 degree from the above discussions. On the other hand, estimated attitude by the constant attitude model, which is marked by 'Δ' in Figure 2, has an uncertainty in determining an attitude epoch. If the middle time of the used data span, i.e., 12:00:00(UT), is considered to be the epoch of the attitude as is usually done in operation, the difference between two estimated attitudes seems to be large. If the beginning of the used data span, however, is the epoch, the difference becomes smaller.

An attitude history obtained by another method is also presented in Figure 2, which is marked as '●'. In this method the attitude drift rate, represented as Fourier series of the Sun right ascension, is estimated by the least square method using the attitudes determined by the constant attitude model. Apart from the epoch, an attitude drift locus of our model is in good agreement with that of this method. The epoch is different almost one day, which seems to be caused by the uncertainty of the constant attitude model in determining the attitude epoch.

4. APPLICATION TO OPERATIONS

Using our scheme, we can improve attitude keeping operations in the keeping phase. A standard attitude keeping procedure is shown in Figure 3. In this procedure attitudes are determined by the constant attitude model using one day data. Errors due to the attitude determination model are absorbed by shortening the operation rest period. On the other hand, new procedure using the attitude drift model, which is shown in Figure 4, can bring us much operation rest period and simplify the operation procedure. In particular, the new procedure is efficient in the case that the required region of attitude keeping is narrow.

This new procedure is planned to be applied to the future Communications Satellite-3 (CS-3a and CS-3b), which are scheduled to be launched in 1988

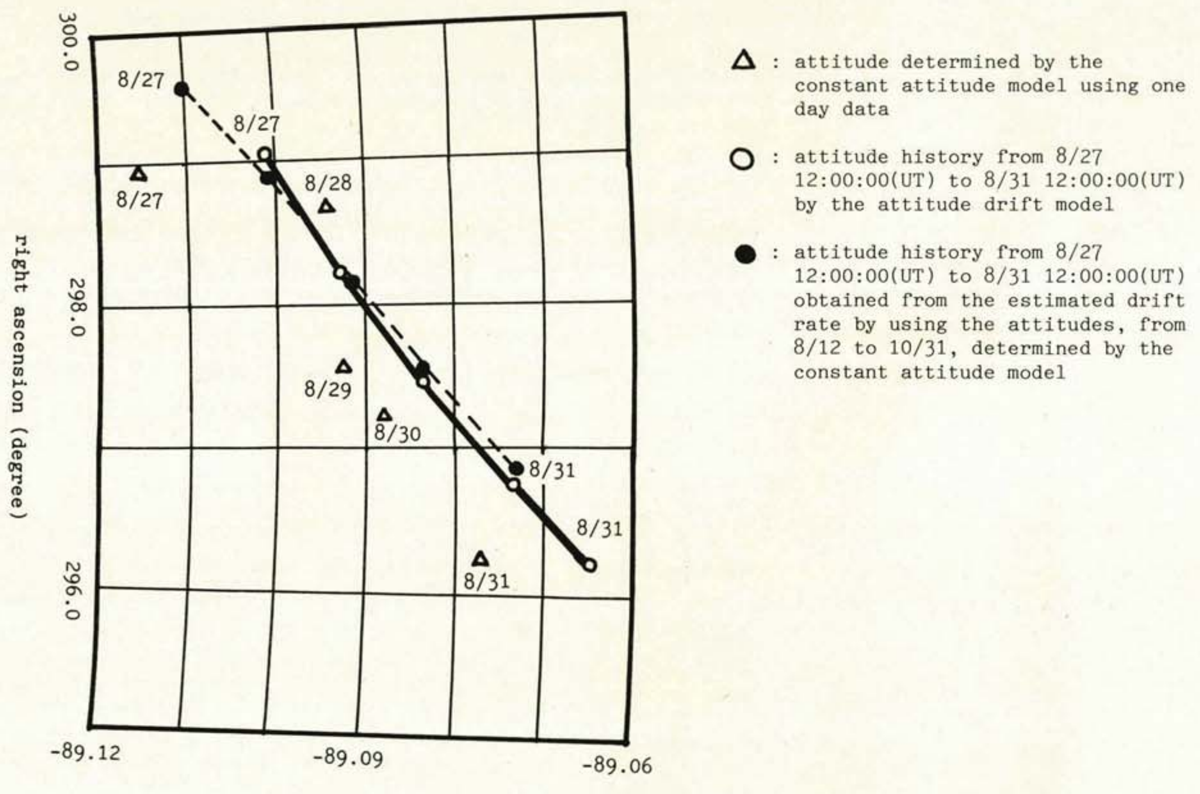


Figure 2. Comparison with other results

by the H-I launch vehicle of NASDA. The attitude of CS-3 has to be kept within the tolerance of 0.055 degree from the anti-orbit-normal direction in order to achieve the mission requirement. This severe requirement has never been experienced by NASDA so far, and our estimation scheme will much help the satellite operation in the keeping phase.

5. CONCLUDING REMARKS

An attitude determination scheme in the keeping phase based on the attitude drift model has been presented. Besides our drift model, many other drift models may be assumed. For example, the model with periodically changing drift rate and data biases with respect to time can be considered. Furthermore, the earth widths can be used as measurements in the estimation process. These were examined by Mr. Tohchi and Mrs. Aoki of Fujitsu Ltd. Noticeable results, however, can not be obtained compared with our model.

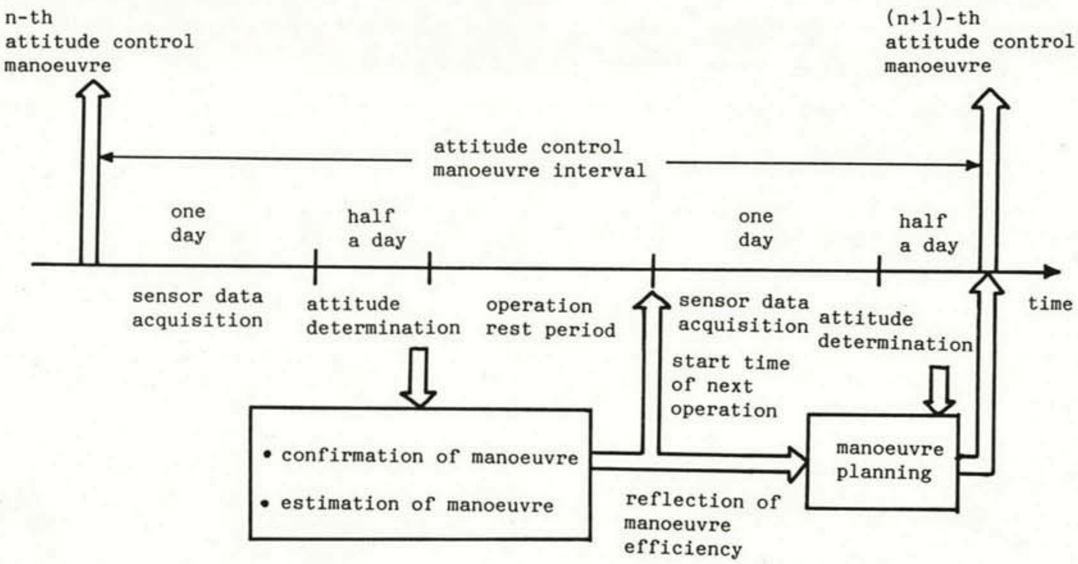


Figure 3. Standard attitude keeping procedure

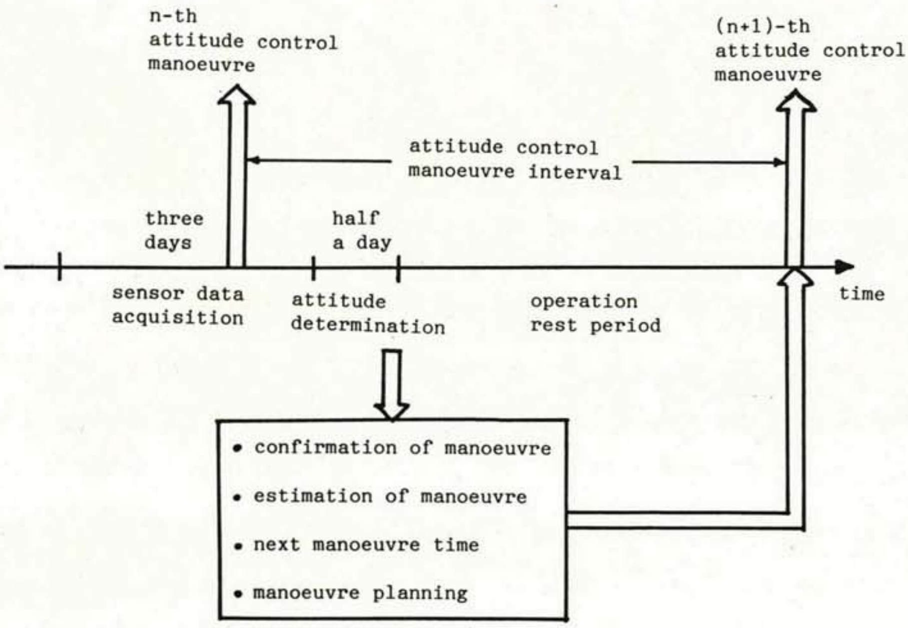


Figure 4. Attitude keeping procedure by the constant attitude model

6. REFERENCES

1. Wertz, J. R., ed., 1978, Spacecraft Attitude Determination and Control, Dordrecht, D. Reidel Publishing Co.

2. Horii, M., 1985, Attitude Determinations for Geostationary Satellite of NASDA, Proceedings of Space Dynamics for Geostationary Satellites, CNES, Toulouse.

3. Meyer, W., Fraiture, L., Wiengarn, N., 1985, Coverage Transition Times of IR-Pencil Beam Sensors, Proceedings of Space Dynamics for Geostationary Satellites, CNES, Toulouse.

4. Van der Ha, J. C., 1986, Attitude Determination Covariance Analysis for Geostationary Transfer Orbit, J. Guidance 9(2), 156-163.

5. Horii, M., 1985, A Procedure of Global Analyses for Spacecraft Attitudes, NASDA inner report TK-M052013.

APPENDIX: GLOBAL ANALYSIS PROCEDURE FOR SPIN-AXIS ATTITUDE

Usually the necessity of derivatives occurs in a 'local' analysis. Then any local coordinates, e.g., the pair of right ascension and declination, have been sufficient for the application. However, the reference or centering point $\mathbf{a}$ itself is of 'global' concern. In this mean, if possible, a 'global' analysis procedure is most desirable. Here we show a simple and efficient representation of derivatives for functions on S^2 with some examples (Ref. 5).

Let $f(\mathbf{a})$ be functions of class C^1 defined on (an area of) $S^2 = \{\mathbf{a} \in R^3 ; \mathbf{a}\cdot\mathbf{a} = 1\}$. Then, the efirst-order derivative of $f(\mathbf{a})$, $\nabla f(\mathbf{a})$, is defined as

$$f(\mathbf{a}) = f(\mathbf{a}_0) + \nabla f(\mathbf{a}_0)\cdot(\mathbf{a} - \mathbf{a}_0) + o(|\mathbf{a} - \mathbf{a}_0|) \quad \text{(A-1)}$$

Here, $\nabla f(\mathbf{a}_0)$ is a three-dimensional vector tangential to S^2 at $\mathbf{a} = \mathbf{a}_0$, and $\nabla f(\mathbf{a}_0)\cdot(\mathbf{a} - \mathbf{a}_0)$ is the inner product of two vectors. Derivatives of this definition as vectors are successfully applicable to attitude analysis.

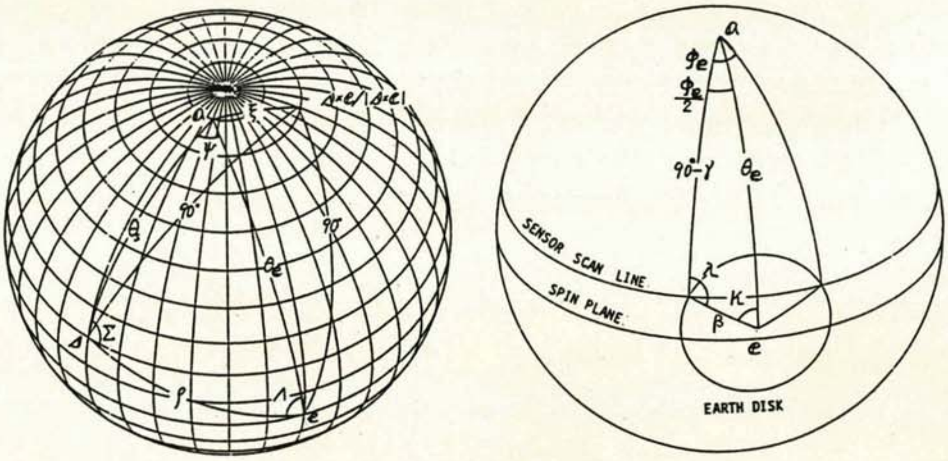
The calculation procedure for $\nabla f(\mathbf{a}_0)$ is as follows. A function $f(\mathbf{a})$ on S^2 is smoothly extended to a function $\tilde{f}$ of R^3, i.e.,

$$\begin{aligned} f(\mathbf{a}) &= \tilde{f}(a_1, a_2, a_3) \\ &= \tilde{f}(\mathbf{a}\cdot\mathbf{i}, \mathbf{a}\cdot\mathbf{j}, \mathbf{a}\cdot\mathbf{k}) \end{aligned} \quad \text{(A-2)}$$

Then, the derivative $\nabla f(\mathbf{a})$ is calculated by the chain rule.

$$\nabla f(\mathbf{a}) = \frac{\partial \tilde{f}}{\partial a_1}\nabla(\mathbf{a}\cdot\mathbf{i}) + \frac{\partial \tilde{f}}{\partial a_2}\nabla(\mathbf{a}\cdot\mathbf{j}) + \frac{\partial \tilde{f}}{\partial a_3}\nabla(\mathbf{a}\cdot\mathbf{k}) \quad \text{(A-3)}$$

Thus, it is sufficient to show a calculation of $\nabla(\mathbf{a}\cdot\mathbf{u})$, where $\mathbf{u}$ is a constant vector, as



$\mathbf{s}$: sun direction
$\mathbf{e}$: earth center direction
β : earth disk half view angle
γ : earth sensor mounting angle

Figure 5. Geometry of spin-axis attitude observations

$$\nabla(\mathbb{a}\cdot\mathbb{u}) = \mathbb{u} - (\mathbb{a}\cdot\mathbb{u})\mathbb{a} \qquad \text{(A-4)}$$

An efficient abbreviation notation is as follows. A derivative $\nabla f(\mathbb{a})$ is known to be tangential to S^2 at $\mathbb{a} \in S^2$. Then, $\nabla f(\mathbb{a}) + \varepsilon\mathbb{a}$ with any scalor ε can represent $\nabla f(\mathbb{a})$, and the notation '$\sim$' is introduced on behalf of '='. Then, we can write

$$\nabla(\mathbb{a}\cdot\mathbb{u}) \sim \mathbb{u} \qquad \text{(A-5)}$$

Some examples are shown. Let the notations be as Figure 5. The following observation equations are given.

sun angle

$$\eta_s = \sin^{-1}(\mathbb{a}\cdot\mathbb{s})$$

nadir angle

$$\theta_e = \cos^{-1}(\mathbb{a}\cdot\mathbb{e})$$

(A-6)

earth width

$$\phi_e = 2\cos^{-1}\left\{(\cos\beta - \sin\gamma\,\mathbb{a}\cdot\mathbb{e})/(\cos\gamma\sqrt{1-(\mathbb{a}\cdot\mathbb{e})^2})\right\}$$

dihedral angle

$$\psi = \tan^{-1}\left\{\mathbb{a}\cdot(\mathbb{s}\times\mathbb{e})/(\mathbb{s}\cdot\mathbb{e} - (\mathbb{a}\cdot\mathbb{s})(\mathbb{a}\cdot\mathbb{e}))\right\}$$

Then, the derivatives of these functions are calculated as

$$\nabla\eta_s \sim (\cos\eta_s)^{-1}\mathbb{s}$$

$$\nabla\theta_e \sim -(\sin\theta_e)^{-1}\mathbb{e}$$

(A-7)

$$\nabla\phi_e \sim 2(\tan\kappa\,\sin^2\theta_e)^{-1}\mathbb{e}$$

$$\nabla\psi \sim (\sin\theta_s\sin\theta_e)^{-1}\left\{\sin\psi(\cos\theta_e\mathbb{s} + \cos\theta_s\mathbb{e}) + \cos\psi\,\mathbb{s}\times\mathbb{e}\right\}$$

When the vector nature, i.e., row or column, of the derivative should be specified, it is understood as a row vector.