A NEW SET OF CONICAL VARIABLES FOR ORBIT CALCULATION

J M Ferrándiz

Dpt. Matemática Aplicada a la Técnica
E.T.S.I.I., 47011 VALLADOLID
SPAIN

ABSTRACT

In this article we present a new canonical transformation which is useful to describe the orbital motion of a particle, and involves redundant variables. We show that it shares interesting properties with the KS transformation. When the perturbation comes from the oblateness of the primary body, it seems reasonable to use the canonical equations , because their right hand parts result into polynomial expressions. In any case, we can use the transformation to reduce the two-body problem to four harmonic oscillators. Even more, their common frequency is equal to one if we use a new time coincident with the true anomaly. We think that this transformation could be used as an alternative method for an efficient calculation of orbits.

Keywords: Canonical transformation, KS variables linearization methods.

1. INTRODUCTION

The use of the KS transformation is a proper way to get high accuracy solutions for orbital problems. We refer only to some of the many properties of this transformation which are extensively treated in the book of Stiefel and Scheifele (Ref. 1). If the orbit is a perturbed ellipse and we introduce a time proportional to the eccentric anomaly , the KS transformation reduces the two-body problems to four perturbed harmonic oscillators. Even more, the canonical KS version allow us to take advantage of the Hamiltonian formalism.

It would be highly desirable to have a canonical transformation to reduce any kind of orbital motion to perturbed oscillators, without losing certain properties of the KS-transformation. Such is the main purpose of this paper.

We first define a transformation from the Cartesian canonical variables to a set of eight redundant variables which we prove is canonial. The definition of the new coordinates is very simple, since they can be considered as homogeneous Cartesian coordinates in a projective space. The expansions of the potential are far more simpler in our case than if one uses the KS variables. However, the definition of the momenta turns out to be complicated, and the transformed Hamiltonian is not a harmonic oscillator one. As it happens with the KS transformation, the new variables satisfy a sort of bilinear relationship that may be used as a check in the numerical integration of the canonical equations.

In the third section we introduce a ficticious time proportional to the true anomaly. In this way we obtain canonical equations that have polynomial rigth hand sides when the potential is due to the oblateness of the primary. The problem, formulated in the new variables, has three first integrals, which can be used to transform the canonical equations in many possible ways. This section ends up with a geometric interpretation of these variables.

Further on, we derive second order equations for the coordinates, that are four perturbed harmonic oscillators. Their common frequency can be taken either as the magnitude of the angular momentum or as one, if the new time coincides with the true anomaly.

We finish this communication showing that we can modify our canonical set in such a way that the radius becomes a coordinate, and we obtain equations with polynomials in the rigth hand sides if the potential does not contain negative powers of the radial distance.

2. DEFINING A NEW SET OF CANONICAL VARIABLES

Let us consider the motion of a particle under a potential V that depends only on its position. We denote by $X = (X_1, X_2, X_3)$ the location vector, $P = (P_1, P_2, P_3)$ the Cartesian conjugate momenta vector, and $|P|$ the norm of P. The particle's orbit is the solution of the canonical equations for the Hamiltonian.

$$H = \tfrac{1}{2} |P|^2 + \bar{V}(X) \tag{1}$$

Let x_1, x_2, x_3 and z four new coordinates of the particle's position related to the original X by

$$X_i = x_i/z \qquad (i = 1,2,3) \tag{2}$$

Equations (2) define a maximum rank transformation

Proceedings of the Second International Symposium on Spacecraft Flight Dynamics, Darmstadt, FR Germany, 20–23 October 1986 (ESA SP-255, Dec. 1986)

mapping the space of vectors $\underline{x} = (x_1, x_2, x_3, z)$, with $z \neq 0$, on the ordinary space; the pre-image of a point X being the straight line $\{z(X_1, X_2, X_3, 1); z \neq 0\}$. Let us define a vector $\underline{p} = (p_1, p_2, p_3, p_z)$ of new momenta verifying the following relationships

$$P_i = p_i z - \frac{x_i z}{|x|^2} (\underline{x}, \underline{p}) \qquad (i = 1,2,3) \tag{3}$$

where $x = (x_1, x_2, x_3)$ and $(\underline{x}, \underline{p})$ stands for the doot product.

To simplify the simbolisms, we occasionally write z and p_z as x_4 and p_4. The roman indexes will be assumed to range from 1 to 3, the greek ones from 1 to 4, and we use Einstein's convention.

Taking Eqs. (2) and (3) into (1) we get a new function

$$H = \tfrac{1}{2} |p|^2 z^2 + \tfrac{1}{2} \frac{p_z^2 z^4}{|x|^2} - \tfrac{1}{2} \frac{(x,p)^2 z^2}{|x|^2} + V(\underline{x}) \tag{4}$$

where $(x,p) = x_i p_i$, and

$$V(\underline{x}) = \bar{V}(x_1/z,\ x_2/z,\ x_3/z) \tag{5}$$

We shall see later on that using Eqs. (3) we can extend the change of coordinates (2) to a canonical transformation increasing the number of variables, the function (4) being the new Hamiltonian. In fact, when we substitute into Eqs. (2) and (3) a solution of the canonical equations

$$\frac{dx_\alpha}{dt} = \frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial x_\alpha} \tag{6}$$

we get a solution to the original problem, as long as the initial values $\underline{x}^\circ$, $\underline{p}^\circ$ satisfy certain conditions.

Let us prove the following lemmas.

Lemma 1.- The bilinear expression

$$T(\underline{x}, \underline{p}) = (\underline{x}, \underline{p}) \tag{7}$$

is a first integral of the new canonical equations (6).

Proof.- It is sufficient to show that the Poisson bracket $\{T,H\}$, defined in the new variables $\underline{x}$, $\underline{p}$, becomes zero.

Let $H_o = H - V$. A trivial calculation leads to the equality $\{T, H_o\} = 0$.

On the other hand, since the function V was generated from $\bar{V}$ according to the identity (5), it follows

$$\frac{\partial V}{\partial z} = -\frac{x_i}{z^2} \frac{\partial \bar{V}}{\partial X_i}, \quad \frac{\partial V}{\partial x_i} = \frac{1}{z} \frac{\partial \bar{V}}{\partial X_i} \tag{8}$$

From (8) and (9) below

$$\{T, V\} = z \frac{\partial V}{\partial z} + x_i \frac{\partial V}{\partial x_i} \tag{9}$$

we get $\{T,V\} = 0$. This completes the proof.

Lemma 2.- For any set of initial conditions X°, P° in the original problem (1), one can choose a set of initial values $\underline{x}^\circ$, $\underline{p}^\circ$ for the new variables to satisfy

$$T(\underline{x}^\circ, \underline{p}^\circ) = (\underline{x}^\circ, \underline{p}^\circ) = 0 \tag{10}$$

Proof.- Let us relate the new and old initial values for the coordinates by

$$X_i^\circ = x_i^\circ / z^\circ \tag{11}$$

If we take p_i° and p_z° to be

$$p_i^\circ = P_i^\circ / z^\circ, \quad p_z^\circ = -(X^\circ, P^\circ)/z^\circ \tag{12}$$

it is obvious that Eq. (10) holds.

Using the above Lemmas we can prove our main result

Theorem.- Assume that the initial values $\underline{x}^\circ$, $\underline{p}^\circ$ for the new variables. obtained from original initial values X°, P°, satisfy the relation (10). If the corresponding solution $\underline{x}(t)$, $\underline{p}(t)$ of the canonical system (6) is transformed by Eqs. (2) and (3), it will give us the solution of the original canonical system, with the given initial values X°, P°.

Proof.- Functions $\underline{x}(t)$, $\underline{p}(t)$ are the solution of the Pfaff's system associated with the form

$$\omega = p_\alpha dx_\alpha - H(\underline{x}, \underline{p}) dt \tag{13}$$

for the initial values $\underline{x}^\circ, \underline{p}^\circ$.

If we substitute into (13) the equations

$$x_i = zX_i, \quad p_i = \frac{P_i}{z}, \quad p_z = -\frac{(X,P)}{z} \tag{14}$$

we get the new expression for ω

$$\omega = P_i dX_i - H(X,P) dt \tag{15}$$

where H turns out to be the Hamiltonian given in Eq. (1).

It is obvious that $\underline{x}(t)$ and $\underline{p}(t)$ are transformed through (14) into a solution of the Pfaff's system associated to the form (15), satisfying

$$X_i(t) = x_i(t)/z(t), \quad P_i(t) = p_i(t) z(t) \tag{16}$$

Functions X(t), P(t) are, in this way, solutions to the system (1). Lemma 1 and condition (10) allow us to write $T(\underline{x}(t), \underline{p}(t)) = 0$ for all t. Accordingly, (16) are precisely the functions that one would get putting the curves $\underline{x}(t)$, $\underline{p}(t)$ into expressions (2) and (3).

Q.E.D.

Finally, let us remark that since T=0 along the solution to Eqs. (6), we can simplify the Hamiltonian (4). Those terms in T^2 can be removed, and we can write the equivalent Hamiltonian K

$$K = \frac{1}{2} |p|^2 z^2 - \frac{(x,p) z^2}{|x|^2} - \frac{(x,p) z^3 p_z}{|x|^2} + V \tag{17}$$

3. EQUATIONS OF MOTION WITH RESPECT TO A FICTICIOUS TIME

Assuming a Newtonian attraction, the potencial V can be split into the Keplerian potential plus a perturbing term U,

$$V(\underline{x}) = -\frac{\mu z}{|x|} + U(\underline{x}) \tag{18}$$

with μ the reduced mass of the particle.

In what follows it would be convenient to introduce the homogeneous formalism. Let x_o be the extra coordinate coinciding with the physical time, and p_o its conjugated moment, with values defined in order to satisfy

$$H_h = H + p_o = 0 \tag{19}$$

in the new homogeneous Hamiltonian H_h.

To simplify the equations of motion in the new variables, we will introduce a suitable ficticious time, proportional to the true anomaly. We choose the new time s as a solution of the differential equation (20)

$$dt = |x|^2 z^{-2} ds \tag{20}$$

The resulting Hamiltonian is

$$\tilde{H}_h = \frac{1}{2}|p|^2|x|^2 + \frac{1}{2}p_z^2 z^2 - \frac{1}{2}(x,p)^2 - $$

$$- \frac{|x|}{z}\mu + W + p_o\frac{|x|^2}{z^2} = 0 \tag{21}$$

with

$$W = |x|^2 z^{-2} U \tag{22}$$

If one uses (.)' for the derivatives with respect to s, the canonical equations from (21) are

$$x_i' = |x|^2 p_i - (x,p)x_i$$

$$z' = z^2 p_z$$

$$x_o' = |x|^2 z^{-2} \tag{23}$$

$$p_i' = -|p|^2 x_i + (x,p)p_i + \frac{\mu x_i}{z|x|} - \frac{\partial W}{\partial x_i} - \frac{2p_o x_i}{z^2}$$

$$p_z' = -z\,p_z^2 - \frac{\mu|x|}{z^2} - \frac{\partial W}{\partial z} + \frac{2p_o|x|^2}{z^3}$$

$$p_o' = 0 \tag{24}$$

Eq. (24) for p_o shows that we are dealing with a conservative system. We use this restriction all along the paper for simplicity, since the main results will still verify without such a restriction.

It is useful to note that the following simple statement holds.

<u>Proposition</u>.- $|x|^2$ is a constant of motion, which can be taken as 1.

<u>Proof</u>.- From Eq. (20) we deduce that $x_i x_i' = 0$.

If we determine $\underline{x}°$, $\underline{p}°$ to satisfy $z° = |X°|^{-1}$, plus relation (9), then $|x|^2 = 1$. This is always available if we exclude the unlikely possibility of colission at the initial time.

With this choice, variable x can be seen as the unitary vector in the direction of the particle, whereas z is the inverse of the radial distance. In the same vein, Eqs. (14) show that p_i has the dimension of an angular momentum. With regard to p_z, the bilinear relationship $T = 0$, plus Eq. (16), lead us to Eq. (14), indicating that p_z is the velocity of variation of z.

We finally point out that the equations of motion (23), (24) can be modified by introducing in their right hand sides some of the first integrals for our problem which we have refered above: $T = 0$, $|x|^2 = 1$ and the energy integral, which can be written as in Eq. (21) or alternatively as

$$\frac{1}{2}c^2 + \frac{1}{2}p_z^2 z^2 - \frac{\mu}{z} + W + \frac{p_o}{z^2} = 0 \tag{25}$$

where c is the norn of the angular momentum. This statement follows from the identity

$$c^2 = |x|^2|p|^2 - (x,p)^2 \tag{26}$$

From the last two integrals, we come to the transformed equations

$$x_i' = p_i - (x,p)x_i$$

$$z' = z^2 p_z$$

$$x_o' = 1/z^2 \tag{27}$$

and either

$$p_i' = -|p|^2 x_i + (x,p)p_i + \frac{\mu x_i}{z} - \frac{\partial W}{\partial x_i} - \frac{2p_o x_i}{z^2}$$

$$p_z' = -zp_z^2 - \frac{\mu}{z^2} - \frac{\partial W}{\partial z} + \frac{2p_o}{z^3} \tag{28}$$

or

$$p_i' = (x,p)p_i - \frac{\mu}{z}x_i - \frac{1}{z^2}\frac{\partial U}{\partial x_i}$$

$$p_z' = -zp_z^2 + \frac{\mu}{z^2} - \frac{1}{z}|p|^2 - \frac{1}{z^2}\frac{\partial U}{\partial x_i} \tag{29}$$

Some other systems are similarly available. The final choice of one of such systems will depend upon proper numerical simulations.

4. REDUCING THE PROBLEM TO THE OSCILLATOR FORM

The canonical equations for the coordinates and the momenta can be substituted alternatively for a set of perturbed second linear equations. In fact, if we substitute Eqs. (23) and (24) into the derivatives of Eqs.(23), and consider the integral $|x|^2 = 1$ and the integral of the energy (25), the following equations are obtained

$$x_i'' + c^2 x_i = -\frac{\partial W}{\partial x_i} + \left(\frac{\partial W}{\partial x}, x\right)x_i \tag{30}$$

$$z'' + c^2 z = \mu - \frac{\partial U}{\partial z} \tag{31}$$

where $\partial W/\partial x$ is the gradient of W with respect to x.

The frecuency of the perturbed oscillators (30) and (31) varies according to the law

$$c' = \frac{1}{c}\left(\frac{\partial W}{\partial x}, x'\right), \tag{32}$$

and is not constant but in rather particular cases.

For instance, in the Keplerian case, W = U =0 and our problem reduces exactly to four harmonic oscillators. The frecuency will be also constant if the potential only depens on the radius.

In the general case, when c is not constant, the user will have to make his choice, and perhaps use Eqs. (27) and (28). They are simpler when the perturbations are due to the oblateness of the primary, since then their right hand sides are polynomials except for a few terms.

We can get alternative but similar equations, in which the frecuency turns out to be the unity. Defining a time v by means of relation

$$dt = \frac{|x|^2}{z^2\ c} dv. \tag{33}$$

it will coincide with the true anomaly.

The reader can obtain those equations by simply noting that $(.)^{\cdot} = (1/c)(.)'$, where $(.)^{\cdot}$ means the derivative with respect to v.

A line of argument similar to the one used at the opening of this section will lead us to the following second order equations

$$\ddot{x}_i + x_i = \frac{1}{c^2}\left[-\frac{\partial W}{\partial x_i} + \left(x, \frac{\partial W}{\partial x}\right)x_i + \right.$$
$$\left. + \left(\dot{x}, \frac{\partial W}{\partial x}\right)\dot{x}_i \right] \tag{34}$$

$$\ddot{z} + z = \frac{1}{c^2}\left[\mu - \frac{\partial U}{\partial z} - \left(\dot{x}, \frac{\partial W}{\partial x}\right)\dot{z}\right] \tag{35}$$

Let us finally comment that all the equations obtained are independent of the type of the orbit, and so are particularly interesting when the trajectory is not necessarily elliptic.

5. AN ALTERNATIVE SET OF EQUATIONS

It is advantageous to have equations of motion with right hand side members being polynomials in the variables, since it will decrease the local error in numerical integrations. In the case of interior three body problems, we can get equations with this property using canonical variables (x, r, p, p_r) obtained from (x,z,p,p_z) by defining

$$r = 1/z \quad , \quad p_r = -p_z/z^2 \tag{36}$$

It is obvious that this change of variables gives a canonical transformation, which leads to the Hamiltonian

$$H_h^* = \frac{1}{2}|p|^2|x|^2 + \frac{1}{2}p_r^2 r^2 - \frac{1}{2}(p,x)^2 -$$
$$- \mu|x|r + W^* + p_o|x|^2 r^2 \tag{37}$$

where

$$W^* = r^2|x|^2 U, \tag{38}$$

and we use the time s introduced in Eq.(20).

The resulting canonical equations are

$$\begin{aligned} x_i' &= p_i - (x,p)x_i \\ r' &= r^2 p_r \\ x_o' &= r^2 \\ p_i' &= -|p|^2 x_i + (x,p)x_i + \mu r x_i - \frac{\partial W^*}{\partial x_i} - 2p_o r^2 x_i \end{aligned} \tag{39}$$

$$p_r' = -rp_r^2 + \mu - \frac{\partial W^*}{\partial r} - 2p_o r \tag{40}$$

They have polynomial right hand side expressions, whenever W* does not contain negative powers of r.

Acknowledgements. The author thanks the financial assistence of CONAI and DGA, under grant No. CB2/85

9. REFERENCES

1. Stiefel, E. & Scheifele, G. 1971, Linear and Regular Celestial Mechanics, Berlin,Springer-Verlag .