Use of Some Volumic Methods in Resolution of Two-Point Boundary Value Problems

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Abstract

An approach to solve optimal control problems is to use Pontryagin's maximum principle. This principle gives optimality necessary conditions, but it needs the resolution of a two-point boundary value problem (TPBVP). This type of differential problems is very difficult to solve because being very sensitive to the bound conditions. Usually, the solution is found by shooting methods which make use Newtonian type techniques, so that some problems may occur when the condition number of Jacobian matrix is poor causing divergence if the starting point is not close to the solution. A way to avoid these problems can be the use of direct global optimisation methods (the ones for which only the cost function values are required) to solve TPBVP, and this is the approach proposed in this paper. After some theoretical issues and the presentation of numerical algorithm, some examples are developed to show the efficiency of the approach.

Key words : two-point boundary value problems, optimal low-thrust interplanetary trajectories.

Introduction

One can in general terms split the optimisation and optimal control methods into two broad categories : the direct methods[1, 2, 3] and the indirect ones. The direct methods can be presented as the ones working on the primal variables and, iteratively adjusting the control variables from an admissible initial guess in an attempt to continuously reduce a performance index. Convergence is not generally a big problem but, off course the convergence is not guaranteed to the global minimum and only a local minimum is achieved. In indirect methods involving additional dual or (costate) variables are, at least theoretically, intended to find the global optimum and avoid some drawbacks linked with the knowledge of an initial feasible solution on the primal variables. However they suffer from an increase in the problem dimension which, in the nonlinear context can cause severe numerical problems.

Pontryagin's Maximum Principle

For instance the Pontryagin's maximum principle, which can be cast in the indirect methods category, needs the solution of a TPBVP, which is known to raise serious difficulties, for instance, in the case when the control variables present some discontinuities.

Let us consider the following control problem :

$$\underset{U}{Min} \int_{t_0}^{t_f} g(X,U,t)dt \; + \; \phi\left(X\left(t_f\right),t_f\right) \qquad (1)$$

$$\dot{X} \; = \; f(X,U,t) \qquad X(t_0) \; = \; X_0 \qquad (2)$$

The necessary optimality conditions are given below after the introduction of the Hamiltonian function (3).

$$H(X,U,\lambda,t) \; = \; g(X,U,t) \; - \; \lambda^T f(X,U,t) \qquad (3)$$

$$\frac{\partial H}{\partial \lambda} = -\dot{X} \quad \Leftrightarrow \quad \dot{X} = f \qquad (4)$$

$$\frac{\partial H}{\partial X} = \dot{\lambda} \qquad (5)$$

where the optimal control U^* must be the solution of the minimisation problem (6),

$$U^* = \underset{U}{Min} \quad H(X^*,U,\lambda^*,t)\,. \qquad (6)$$

The state variable X and the costate variable λ are in R^n the control U is in R^m. X^* and λ^* are the solutions of the differential system made up of (4) and (5). The problem (6) is for an unconstrained control problem, if the control is constrained U must be taken in its definition set $\boldsymbol{U}$. The initial state conditions are given in (2), and in addition, some final dual state are to be settled in order to define solutions for the TPBVP (4), (5). These last conditions are called transversality conditions and act on the costate variables. Depending on the optimisation problem, they are written as :

$$X(t_f) = X_f, \quad t_f \ \ fixed \tag{7}$$

$$H\Big|_{t_f} + \left.\frac{\partial \phi}{\partial t}\right|_{t_f} = 0 \quad and \quad X(t_f) = X_f, \quad t_f \ \ free \tag{8}$$

$$\lambda(t_f) = -\left.\frac{\partial \phi}{\partial X}\right|_{t_f}, \quad t_f \ \ fixed \tag{9}$$

$$H\Big|_{t_f} + \left.\frac{\partial \phi}{\partial t}\right|_{t_f} = 0 \ \ ; \ \ \lambda(t_f) = -\left.\frac{\partial \phi}{\partial X}\right|_{t_f}, \quad t_f \ \ free. \tag{10}$$

The TPBVP is formed by the differential system (4), (5), the initial state conditions (2) and the transversality conditions (7),…, (10). There are no initial conditions for the costate variables, and the principle of our approach is based on searching for good ones in order to respect the optimality necessary conditions, and specially the transversality conditions at the final time.

Resolution of a TPBVP

There exist several methods to solve TPBVP, we can quote three useful types :

- shooting methods,
- integral equation methods,
- finite differences methods.

For lot of problems these methods work very well and give the optimal solution. But some problems occur when discontinuities appear on the control variables or if the function f in (2) is non smooth. The method we propose can be classified in the shooting methods category, because it is an initial-value method[4]. This new approach is shown to have some efficiency to solve TPBVP even if properties of continuity of the control variables are not verified (in this case we can also use continuation methods to approximate the solution). The goal of our method is to view the TPBVP as an optimisation problem in order to find the correct initial costate conditions which permit us to obtain the desired final conditions (7), …, (10). We define the optimisation variables as the unknown initial conditions. If all the initial state conditions are fixed and if the state X is in R^n then the number of optimisation variables is n and they are noted λ^0_i ($i=0, \ldots, n$). Generally, anything is known about these variables, so the optimisation space D is taken rectangular and very large. For instance, we can choose :

$$D = D_1 \times D_2 \times \ldots \times D_n \tag{11}$$

where D_i ($i=0, \ldots, n$) are real intervals, which is obviously a compact set. Now we must define an objective function, a common step in the shooting methods, which needs the definition of shooting functions on which Newtonian techniques are applied. Off course, these shooting functions are built from the transversality conditions, so that their minimum correspond to the fulfilment of these conditions. For instance, in the case of p first final states fixed and the q last ones free ($p + q = n$), let us define :

$$F : R^n \to R^n$$
$$\begin{bmatrix} \lambda_1^0 \\ \lambda_2^0 \\ \vdots \\ \lambda_{n-1}^0 \\ \lambda_n^0 \end{bmatrix} \mapsto F = \begin{bmatrix} \tilde{x}_1(t_f) - x_{1f} \\ \tilde{x}_2(t_f) - x_{2f} \\ \vdots \\ \tilde{\lambda}_{n-1}(t_f) + \left.\frac{\partial \phi}{\partial x_{n-1}}\right|_{t_f} \\ \tilde{\lambda}_n(t_f) + \left.\frac{\partial \phi}{\partial x_n}\right|_{t_f} \end{bmatrix} \tag{12}$$

where $\tilde{x}_i(t)$ and $\tilde{\lambda}_i(t)$ ($i=0, \ldots, n$) are obtained by numerical integration of (4), (5) from X_0 and λ_0.

The aim of shooting methods is to find the root of F. Obviously the following objective shooting function G met the previous conditions :

$$G : R^n \to R$$
$$\begin{bmatrix} \lambda_1^0 \\ \lambda_2^0 \\ \vdots \\ \lambda_{n-1}^0 \\ \lambda_n^0 \end{bmatrix} \mapsto G = F^T F \, . \tag{14}$$

Now the optimisation problem can be formulated, it is a minimisation problem where the variables $\lambda^0_{i=1, \ldots, n}$ do not appear explicitly in the objective function.

$$\underset{\substack{\lambda_i^0 \in D_i \\ i=1,\dots,n}}{Min} \quad G \tag{15}$$

Solving this the TPBVP is equivalent to find the minimum of G for which $G = 0$. The existence of this minimum is proved showing that the TPBVP has a solution. If this solution exists then the specific minimum exists too. Some sufficient conditions have been derived for the existence of the solution of the above problem. They are summarised below. Consider the general system subjected to the general linear two-point boundary conditions,

$$\dot{Y} = \eta(t, Y), \quad a < t < b, \tag{16}$$

$$A.Y(a) + B.Y(b) = \alpha, \tag{17}$$

$$Y(t) \in R^n, \alpha \in R^n, (A, B) \in R^{n\times n} \times R^{n\times n}$$

$$\eta : R^{n+1} \to R^n$$

Can be associated to (16), (17) an initial value problem (18) in the following way,

$$\dot{v} = \eta(t, v), \quad v(a) = s \tag{18}$$

$$\gamma(s) = [A.s + B.v(s, b)] - \alpha = 0 \tag{19}$$

where $v = v(s, t)$ is the solution of the problem (18). Clearly, if s^* is the root of equation (19), we expect :

$$Y(t) = v(s^*, t). \tag{20}$$

Definition I : *Let the function $\eta(t, Y)$ be continuous on the infinite strip $R : a \le t \le b$, with $|Y| < \infty$, the function η is said K-Lipschtizian in Y, uniformly in t, if :*

$$|\eta(t, Y) - \eta(t, Z)| \le K.|Y - Z|. \tag{21}$$

Theorem I : *Let $\eta(t, Y)$ be continuous on the infinite strip $R : a \le t \le b$, with $|Y| < \infty$, and satisfy there a uniform Lipschitz condition in Y. Then the boundary value problem (16), (17) has as many solutions as there are distinct roots $s = s^{(\mu)}$ of equation (19) . These solutions are :*

$$Y(t) = v(s^{(\mu)}, t), \tag{22}$$

the solutions of the initial value problem (18) where the variable s equals $s^{(\mu)}$.

The proof of *theorem I* is given in Keller[4]. In practical examples all the required proprieties are not necessary verified, and it is very difficult to prove that the set of solutions of equation (19) is not empty. *Theorem II* prove existence and uniqueness of a solution. This theorem is a modification of Keller's theorem[4] for particular boundary conditions like the ones generally encountered in optimal control problems. The expression of these are given by (23) where A is a $p \times n$ matrix with rank p, B is a $q \times n$ with rank q, $p + q = n$, α is in R^p and β in R^q.

$$(a)\ A.Y(a) = \alpha, \quad (b)\ B.Y(b) = \beta \tag{23}$$

With no loss in generality, the matrix A can be expressed (I_p, A_1) with I_p is the p-order identity matrix and A_1 is a $p \times q$ matrix. Equalities like (23) are derived if the function ϕ in (2) is linear or quadratic with respect to the state X.

Theorem II *: Let $\eta(t,Y)$ satisfy on $R : a \le t \le b$, $|Y| < \infty$,*

$(a) \quad \eta(t, Y) \quad continuous,$

$(b) \quad \dfrac{\partial \eta_i}{\partial y_j} \quad continuous, \quad i, j = 1, 2, \dots, n$

$(c) \quad \left\| \dfrac{\partial \eta}{\partial Y} \right\|_\infty \le k(t)$

$(d) \quad Q^{-1}\ exists, \quad Q = B.\begin{pmatrix} -A_1 \\ I_q \end{pmatrix}$

$(e) \quad \displaystyle\int_a^b k(t)dt \le \ln\left(1 + \frac{\theta}{m}\right), \quad 0 < \theta < 1,$

$$m = \left\| Q^{-1}.B \right\|_\infty . \left\| \begin{pmatrix} -A_1 \\ I_q \end{pmatrix} \right\|_\infty$$

then the problem (16), (23) has an unique solution whatever α and β.

So we show that under some properties of η the TPBVP has a least a solution, which is obtained at the minimum of the function G for which $G = 0$. Formally, we can express the TPBVP [(2)(5)(7)…(10)] through the function η in the following way :

$$\begin{cases} \dot{\varphi} = \eta(\varphi, t) \\ \varphi = \begin{bmatrix} X \\ \lambda \end{bmatrix}, \quad \eta = \begin{bmatrix} f \\ \dfrac{\partial H}{\partial X} \end{bmatrix}. \\ (2), (7), \dots, (10) \end{cases} \tag{24}$$

Now that all is well define, we can explain the resolution of problem (15). The greatest difficulty comes from the expression of the objective function. Indeed, the function G is not analytic, and it can not be expressed explicitly in terms of the variables $\lambda^0_{i=1,\dots,n}$.

That is why, we use in the resolution principle direct optimisation methods. These methods needs only evaluations of the objective function. The convergence rate is off course slower than the convergence rate of methods using first or second derivatives. But direct methods do not need continuity properties of the objective function. So these methods can solve greater class of problems than the others. There exists a lot of direct methods : genetic algorithms[5], clustering methods[6], multistart methods… We have chosen to use a multistart method. Its principle is to apply a local method with a lot of starting points taken uniformly in the optimisation space. If the number of starting point is sufficiently large then we can consider that the best local minimum found is global. We have used two different local methods : nonlinear simplex[7], multireflection simplex[7]. The first (resp. second) is very efficient when the dimension of the optimisation space is small (resp. large). The second method is an extension of the first, so their principle[7] are equivalent. But this method is the second level of the global resolution of (15). Indeed, the function G can be considered as a succession of two different functions G_1 and G_2. G_1 solves the initial value problem deduced from (24), and G_2 compute the cost associated to function given by G_1. So we can write :

$$\begin{array}{lcccc} G(\lambda^0_{i=1,\dots,n}) = G_2(G_1(\lambda^0_{i=1,\dots,n})) & & & & \\ G & : & R^n & \rightarrow & R \\ G_1 & : & R^n & \rightarrow & F(R^{2n},R^{2n}) \\ G_2 & : & F(R^{2n},R^{2n}) & \rightarrow & R \end{array} \tag{25}$$

where $F(R^{2n}, R^{2n})$ is the space of functions from R^{2n} to R^{2n}. The function G_2 is not complicated because it just takes the value of the function given G_1 by at the time t_f and produces easily the evaluation of G. So to solve the initial value problem defined from (24), we use numerical integration method : a Runge-Kutta method[4] (RKM). The order of the RKM depends directly on the precision of integration required. Usually, the fourth-order is used. Finally, for each evaluation of the objective function, a numerical integration is done, so the computing time of the global optimisation may be quite high.

To summarise, the method we propose is made up of two levels : the first consists in a numerical integration and computing the associated cost, the second level is to solve the problem (15) with the evaluations given by the first level. The algorithm can be written :

1. chose randomly in D a starting point $\lambda^0_{i=1,\dots,n}$
2. apply the local optimisation method : nonlinear or multireflection simplex, in which evaluations of G_1 and G_2 are used
3. if a local minimum of G is obtained then store it
4. If the starting point number does not equal a number fixed by the user then return to 1. else end.

All the local minima must be analysed, if and only if one of them has a *0* value then the problem (15) is solved, else the algorithm must be restarted with a greater number of starting points. If the *theorem II* is verified, or if only the existence of a solution is proved by *theorem I*, then a sufficiently large number of starting points exists.

For each example of the next part, the number of starting points used and the value of the minimum obtained will be presented.

Numerical examples

In this part three examples will be solved. The first is taken from an article of Herman and Conway[8], the second is a modification of the first, and the final example is a problem of minimum consumption for a low-thrust Earth-Mars transfer. We have chosen these three examples in an increasing order of complexity to show the efficiency of our algorithm.

The first example is an orbit transfer problem in which the rocket engine provides a constant acceleration to the spacecraft. Motion is confined in a single plane. The spacecraft is described with (r, θ) the polar coordinates, the origin is located at the centre of mass of the attracting body. The only control variable is the thrust angle β is measured relative to the local horizontal. Finally, the mass of the spacecraft is constant and A represents the thrust divided by the mass. The state variables are $X = \{r, \theta, v_r, v_t\}^T$ and the evolution system is given by :

$$\begin{array}{lcl} \dot{r} & = & v_r \\ \dot{\theta} & = & \dfrac{v_t}{r} \\ \dot{v}_r & = & \dfrac{v_t^2}{r} - \dfrac{\mu}{r^2} + A.\sin(\beta) \\ \dot{v}_t & = & -\dfrac{v_r.v_t}{r} + A.\cos(\beta) \end{array} . \tag{26}$$

We normalise length (LU) and time units (TU) with which the gravitational constant μ is unity. The acceleration magnitude A, the initial time t_0, and the final time t_f are chosen to be *0.01*, *0.0*, and *50.0* respectively. For this problem there is no integral term in the cost (1), and the expression of the function ϕ is (it represents the specific energy at the final time) :

$$\phi(X(t_f)) = -\left\{\frac{1}{2}\left[v_r^2(t_f)+v_t^2(t_f)\right]-\frac{1}{r(t_f)}\right\}. \qquad (27)$$

Finally, the initial conditions constraints for the state correspond to a circular orbit at a radius of 1.1 LU, resulting in $X_0 = \{1.1, 0.0, 0.0, 1/\sqrt{1.1}\}^T$. For this system the costate variables are $\lambda = \{\lambda_r, \lambda_\theta, \lambda_{vr}, \lambda_{vt}\}^T$, and the expression of the Hamiltonian is :

$$\begin{aligned} H = \lambda_r . v_r + \lambda_\theta . \frac{v_t}{r} + \lambda_{v_r} . \left[\frac{v_t^2}{r} - \frac{1}{r^2} + A.\sin(\beta)\right] \\ + \lambda_{v_t} . \left[-\frac{v_r . v_t}{r} + A.\cos(\beta)\right] \end{aligned}. \qquad (28)$$

The costate equations given by (5) become :

$$\begin{aligned} \dot{\lambda}_r &= \lambda_\theta . \frac{v_t}{r^2} + \lambda_{v_r} . \left[\frac{v_t^2}{r^2} - \frac{2}{r^3}\right] - \lambda_{v_t} . \frac{v_r . v_t}{r^2} \\ \dot{\lambda}_\theta &= 0 \\ \dot{\lambda}_{v_r} &= -\lambda_r + \lambda_{v_t} . \frac{v_t}{r} \\ \dot{\lambda}_{v_t} &= -\lambda_\theta . \frac{1}{r} - \lambda_{v_r} . \frac{2.v_t}{r} + \lambda_{v_t} . \frac{v_r}{r} \end{aligned} \qquad (29)$$

The final costate conditions are given by :

$$\lambda(t_f) = \left\{-\frac{1}{r^2(t_f)}, 0, -v_r(t_f), -v_t(t_f)\right\}^T. \qquad (30)$$

We conclude immediately that for this problem the equalities (23) are not verified because the function ϕ is not linear or quadratic with respect to the state X. So we can not use the *theorem II*. But the variable r is assumed to be great than *1.0*, because *1.0* is the attracting body radius. The magnitude of the spacecraft velocity is assumed to be less than the velocity obtained when r equals *1.0* plus the velocity due to the thrust. So the states function f is continuous and bounded when the thrust angle is assumed to be in *] -π/2 ; -π/2 [*. The expression of this angle is :

$$\tan(\beta) = \frac{-\lambda_{v_r}}{-\lambda_{v_t}}. \qquad (31)$$

Under this hypothesis the function associated with the costate variables is continuous on the interval of evolution. Then the global function η verifies properties of continuity, so we can try to find a solution but the uniqueness of it is not proved. The objective function G is :

$$G(\lambda^0_{i=1,\ldots,3}) = \left\{\begin{matrix} \left(\lambda_r + \frac{1}{r}\right)^2 + (\lambda_{v_r} + v_r)^2 \\ + (\lambda_{v_t} + v_t)^2 \end{matrix}\right\}\Bigg|_{t_f} \qquad (32)$$

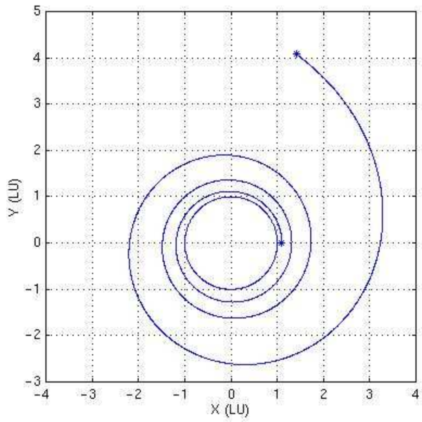

Figure 1 : Optimal trajectory

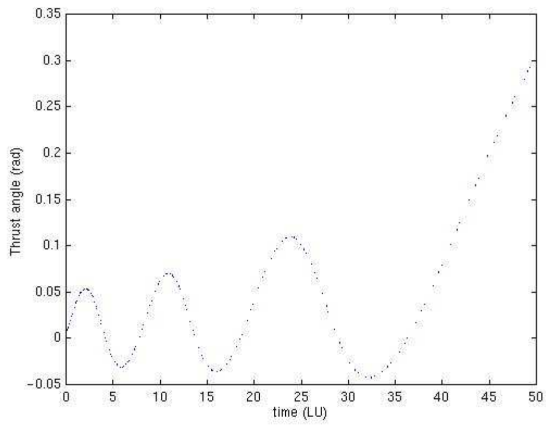

Figure 2 : Optimal thrust angle

There are only three optimisation variables because the final costate conditions imply directly that λ_θ equals *0* for all *t* in $[\, t_0 \,;\, t_f \,]$. The optimisation space is given by (33) and the number of starting points is *50*.

$$D = \left[-10^3 \,; 10^3\right] \times \left[-10^3 \,; 10^3\right] \times \left[-10^3 \,; 10^3\right]. \tag{33}$$

The nonlinear simplex have found *33* times the same minimum and its value is $2.07.10^{-16}$. The optimal trajectory is represented in Figure 1 and the optimal thrust angle in Figure 2. Table 1 gives initial and final state and costate conditions.

Table 1 : Initial and final conditions

	$t_0 = 0.0$	$t_f = 50.0$
r	*1.1*	*4.316*
θ	*0.0*	*20.086*
v_r	*0.0*	*0.156*
v_t	*0.953*	*0.498*
λ_r	*-0.436*	$-5.36.10^{-2}$
λ_θ	*0.0*	*0.0*
λ_{vr}	$-2.32.10^{-3}$	*-0.156*
λ_{vt}	*-0.523*	*-0.498*

The next example is a modification of the preceding one. We solve this example to show the efficiency of the algorithm even if the optimal trajectory is a spiral with many revolutions. So the position angle θ will take very high values. The new values of the constant *A* and t_f are *0.001* and *500.0* respectively. The problem is that the spacecraft must reach a circle with a fixed radius in a finite time. All the differential equations and the initial conditions are not modified and the final conditions become :

$$\begin{aligned} X(t_f) &= \{4.0,\ free,\ free,\ free\} \\ \lambda(t_f) &= \{\ free, 0.0, 0.0, 0.0\} \end{aligned} \quad . \tag{34}$$

With the same number of starting as before, the optimisation space has to be reduced because the problem is very sensitive. The expression of the optimisation space is chosen :

$$D = [-1.0\,; 1.0] \times [-1.0\,; 1.0] \times [-1.0\,; 1.0]\,. \tag{35}$$

A local minimum has been found and its value is : $7.7.10^{-7}$. Figure 3 represents the optimal trajectory and Table 2 the initial and final conditions. This example shows that our approach is able to find the solution of this optimal control problem even if the optimal trajectory presents many revolutions.

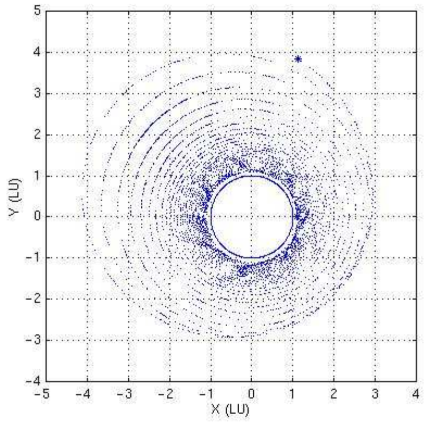

Figure 3 : Optimal trajectory

Table 2 : Initial and final conditions

	$t_0 = 0.0$	$t_f = 500.0$
r	*1.1*	*4.0*
θ	*0.0*	*202.347*
v_r	*0.0*	*0.141*
v_t	*0.953*	*0.470*
λ_r	$-8.1.10^{-4}$	$-9.77.10^{-5}$
λ_θ	*0.0*	*0.0*
λ_{vr}	$-1.0.10^{-4}$	$-7.16.10^{-4}$
λ_{vt}	$-1.0.10^{-4}$	$-5.05.10^{-4}$

Remark 1 : for this example the boundary conditions can be written like conditions (23). Unfortunately, the matrix *Q* (*theorem II*) associated to them is singular. So the uniqueness of a solution is not proved.

The last example presented is a low-thrust Earth-Mars transfer with minimum consumption criterion. This example is given in Coverstone-Caroll and Williams[3]. This problem is a two-body problem (spacecraft - Sun). The spacecraft has a nuclear propulsion system with a constant power source of *450 kW*. The spacecraft engine has a constant specific impulse *c* of *4860 s*. The initial mass is *10000 kg*. The spacecraft departed on Earth on November 19, 1994 with Earth's position and orbital velocity. A rendezvous with Mars occurs *184* days later on May 22, 1995. The state variables are the position *r* in R^3, the velocity *v* in R^3 and the mass *m* in R. The control variables are *U* in R^3, a unit vector defining the thrust direction and δ a variable representing the engine state, δ equals *1* if the engine is on, *0* else.

$$\|U(t)\|_2 = 1, \quad \delta(t) = \begin{cases} 0 \\ 1 \end{cases}, \quad \forall t \in [t_0 ; t_f]. \tag{36}$$

The state differential equations are :

$$\begin{array}{rcl} \dot{r} & = & v \\ \dot{v} & = & -\dfrac{\mu}{\|r\|_2^3}.r + \dfrac{T}{m}.\delta U \\ \dot{m} & = & -\dfrac{T}{c}.\delta \end{array} . \tag{37}$$

The constant T is the thrust magnitude computed with the power source and the specific impulse. μ is the sun gravitational parameter. The problem can be formulated as :

$$\underset{\textit{subject to } (36),(37)}{Min} \; -m(t_f) . \tag{38}$$

The costate differential system is obtained :

$$\begin{array}{rcl} \dot{\lambda}_r & = & \dfrac{\mu}{\|r\|_2^3}.\lambda_v - \dfrac{3}{\|r\|_2^5}.(\lambda_v^T.r).r \\ \dot{\lambda}_v & = & -\lambda_r \\ \dot{\lambda}_m & = & \dfrac{T}{m^2}.(\lambda_v^T.U).\delta \end{array} \tag{39}$$

and the expression of the optimal control :

$$\begin{array}{l} (U^*, \delta^*) = \begin{cases} \left(\dfrac{\lambda_v}{\|\lambda_v\|_2}, 0 \right) & \textit{if } \rho > 0 \\ \left(\dfrac{\lambda_v}{\|\lambda_v\|_2}, 1 \right) & \textit{else} \end{cases} . \\ \rho = \dfrac{T}{c}.\lambda_m - \dfrac{T}{m}.\|\lambda_v\|_2 \end{array} \tag{40}$$

The optimal control is piecewise continuous, that makes the problem very difficult to solve. The number of optimisation variables is *7*, and the expression of the optimisation space is :

$$D = [-10^2 ; 10^2]^3 \times [-10^3 ; 10^3]^3 \times [-1.0 ; 1.0]. \tag{41}$$

With a number of starting points of *500*, the value of the minimum found for G is $2.37.10^{-13}$, so we can conclude that the corresponding point is indeed the solution of problem (38). Figure 4 represents the optimal trajectory in the ecliptic plane, the engine state is shown in Figure 5 and finally, the mass variation is presented in Figure 6.

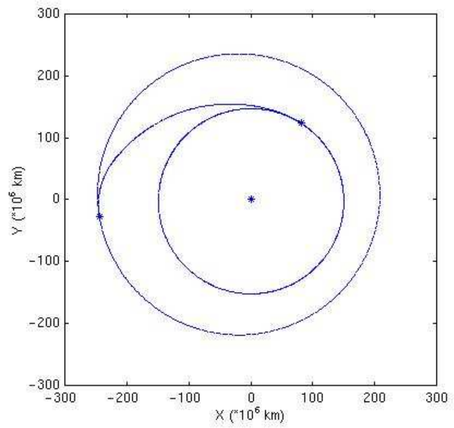

Figure 4 : Optimal trajectory in the ecliptic plane

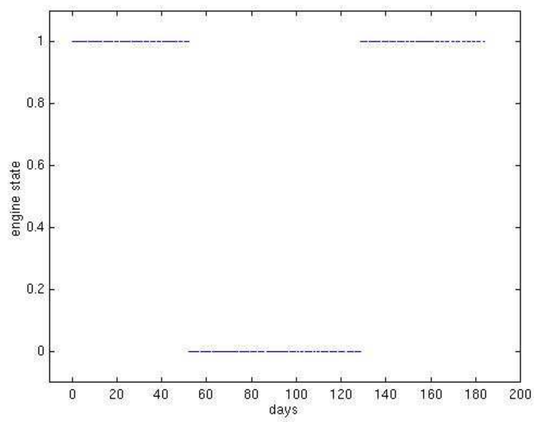

Figure 5 : engine state

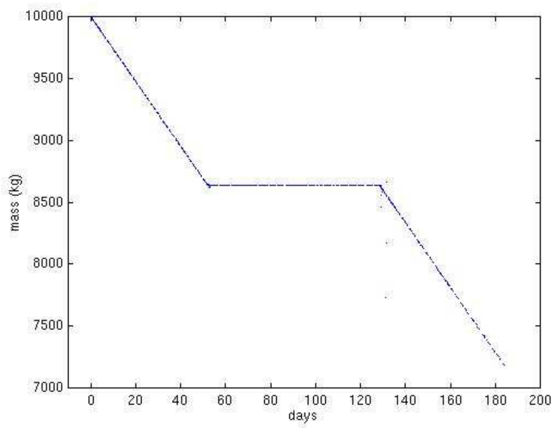

Figure 6 : mass variation

Table 3 presents the state and costate initial and final conditions. The units are 10^6 *km* for the position and 10^6 *km/day* for velocity, the mass is expressed in *kg*.

Table 3 : Initial and final conditions

	11.19.1994	*05.22.1995*
r_x	*81.62*	*-244.14*
r_y	*123.28*	*-27.53*
r_z	$1.52.10^{-3}$	*5.43*
v_x	*-2.19*	*0.31*
v_y	*1.41*	*-1.90*
v_z	$1.41.10^{-5}$	$-4.75.10^{-2}$
m	*10000.0*	*7184.05*
λ_{rx}	*2.18*	*-50.09*
λ_{ry}	*76.35*	*21.68*
λ_{rz}	*-1.35*	*5.07*
λ_{vx}	*-2008.75*	*3716.4*
λ_{vy}	*4046.58*	*-1438.58*
λ_{vz}	*361.22*	*-351.96*
λ_m	*0.55*	*1.0*

The optimal scenario is composed by three periods : the first is a burning period (the engine is on), during the second the engine is off, and the last is a burning period. The duration of the total burning period is not predominant then we can conclude that the transfer time is greater than the minimum transfer time. We find exactly the same results than Coverstone-Caroll and Williams[3] (VARITOP results[3]). To find these results the value of the thrust efficiency variable (it permits to compute the thrust magnitude) is taken to *0.761014*. For this example the results have been found with the nonlinear simplex, but the multireflection simplex can be used too. Finally, to prove the existence and the uniqueness of a solution, we can apply the *theorem II* on each part where the control is continuous defining formally a set of TPBVP.

Remark 2 : notice that the optimal point is not in the set (41). In fact only the starting points must be in, and the search is extended outside the initial research domain.

Conclusions

An optimisation approach for computing solutions of a large range of TPBVPs has been developed. This technique consists in considering the initial costate variables as the optimisation variables in an optimisation problem defined from the transversality conditions. To solve it, direct local optimisation methods are used (only cost function evaluations are required) so solutions can be found even if the control variables present discontinuities. A multistart technique permits to try to find the global optimum and to consider a large optimisation space what does not need the a priori knowledge of the region of convergence of the optimum. Theoretical issues about sufficient conditions have been also presented but it is very difficult to verify them in practical cases, because functions in the differential models do not respect the desired properties of continuity and boundedness. The three examples developed show the efficiency of the approach on different categories of problems. Indeed, with a sufficiently large number of starting points, for each example the global optimum has been found.

Interplanetary low-thrust transfer problems form a very interesting class of optimal control problems. Indeed, the example presented in this paper is a single transfer between two planets with a fixed transfer time. More complex trajectories can be analysed : with the use of planets gravity, free initial and final times which needs a very precise analyse of the planets ephemeris… Future numerical experiments of this approach will be done on this type of complex trajectories.

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