TRANSFER TO THE COLLINEAR LIBRATION POINT L$_3$ IN THE SUN-EARTH+MOON SYSTEM

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Abstract: The collinear libration point L3 of the sun-earth+moon system is an ideal place for some space missions. Although there has been a great amount of work concerning the applications of the other two collinear libration points L1 and L2, little work has been done about the point L3. In this paper, the dynamics of the libration points was briefly introduced first. Then a way to transfer the spacecraft to the collinear libration point L3 via the invariant manifolds of the other two collinear libration points was proposed. Theoretical works under the model of circular restricted three-body problem were done. For the sun-earth+moon system, this model is a good approximation. The results obtained are useful when a transfer trajectory under the real solar system is designed.

Key Words: restricted three-body problem, collinear libration point, invariant manifold

1. Introduction

It is well known that there are three collinear libration points in the circular restricted three-body problem. For the sun-earth+moon system, the collinear libration points L1, L2 are close to the earth. The point L3 is at the other side of the sun, far away from the earth. There has been a lot of work concerning the applications of the points L1 and L2 (see, for example Ref. [1], [2], [3]), but little work has been done about the point L3. The reason partially lies in the large distance between the point L3 and the earth. In fact, the constant thermal conditions and the fixed positions with the sun and the earth make the collinear libration point L3 an ideal place for some space missions (for example, constant observation of what is happening at the back of the sun).

However, the large distance between the point and the earth makes the transfer of and communication with the spacecraft difficult. Different from the collinear libration points L_1 and L_2, the instability of L_3 is very weak. The stable or unstable invariant manifolds may spend thousands of years to leave the proximity of L_3 and can not closely approach the earth. We can not transfer the spacecraft to the invariant manifolds of the nominal orbits as we did in the case of L1 or L2 [4]. Nevertheless, a way to use the invariant manifolds of the point L1 or L2 to transfer the spacecraft to the point L3 was proposed. A relay station around the point L5 (or L4) was used for communication. The way to transfer the relay station was also discussed. Prado has explored the transfer trajectory to the point L3 and back to the earth [5], but there the invariant manifolds were not used.

In section 2, the dynamics of the libration points was briefly introduced. In order to show the differences between the three-body problem and the real solar system, some results under the real gravitations of the solar system were given. In section 3, the ways to find transfer orbits to L3 via the unstable invariant manifold of L2 were given. Some numerical simulations under the circular restricted three-body problem were given. In the conclusion section, some discussions were made.

2. Dynamics of the Libration Points

The restricted three-body problem describes the motion of a massless small body under the gravitations of two massive bodies (called primaries). The usual synodic coordinate which rotates with the two primaries is shown in Figure 1. P_1 and P_2 are the two primaries. C is the barycenter of P_1 and P_2.
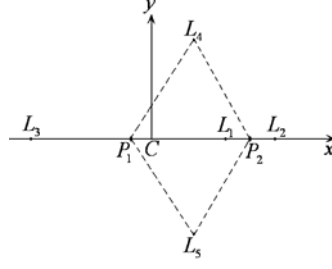
Figure 1. The synodic coordinate and the five libration points

The motion of the small body is [6]

$$\begin{cases} \ddot{\vec{r}} + 2\left(-\dot{y}, \dot{x}, z\right)^T = \left(\partial\Omega/\partial\vec{r}\right)^T \\ \Omega\left(x, y, z\right) = \left(x^2 + y^2\right)\big/2 + \left(1-\mu\right)\big/r_1 + \mu/r_2 \end{cases} \tag{1}$$

Where $\mu = m_2\big/\left(m_1 + m_2\right)$ is called the mass ratio (or mass parameter), with m_1 and m_2 indicating the masses of the two primaries. There is an integral for this system

$$2\Omega - v^2 = 2\Omega - \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = C \tag{2}$$

Where C is the Jacobi constant. There are five equilibrium points in this system, labeled as L_i, ($i = 1, \cdots, 5$) in Figure 1. L_1, L_2, L_3 are collinear libration points and L_4, L_5 are equilateral ones.

Denoting $\left(\xi, \eta, \zeta\right)^T$ as deviations of the small body from the libration points, the linearized motion around these points are

$$\begin{cases} \ddot{\xi} - 2\dot{\eta} = \Omega^0_{xx}\xi + \Omega^0_{xy}\eta + \Omega^0_{xz}\zeta \\ \ddot{\eta} + 2\dot{\xi} = \Omega^0_{yx}\xi + \Omega^0_{yy}\eta + \Omega^0_{yz}\zeta \\ \ddot{\zeta} = \Omega^0_{zz}\zeta \end{cases} \tag{3}$$

Where Ω^0_{**} are constants of the mass ratio μ. It is obvious that the motion of the $x-y$ plane and that of the z axis are decoupled. Since $\Omega^0_{zz} < 0$, the out-of-plane motion of the small body is always stable libration. Denote the eigenvalues of the motion in the $x-y$ plane as $\pm d_1$ and $\pm d_2$. For the collinear libration points, either d_1 or d_2 is a real number. The motion around the collinear libration points is unstable. For the triangular libration points, both d_1 and d_2 are imaginary numbers when $\mu < 0.0385...$, Thus the motion around the equilateral libration point is stable.

2.1 Dynamics of the collinear libration points

Although the motion around the collinear libration points is unstable, there are periodic orbits around them [7]. Two important kinds are the Lyapunov planar orbits and the halo orbits. The Lyapunov orbits lie in the $x-y$ plane. They are symmetric with the x axis and have two perpendicular intersections with the x axis. The halo orbits are periodic orbits in 3D space. They are symmetric with the $x-z$ plane and have two perpendicular intersections with the $x-z$ plane. Besides the periodic orbits, there are quasi-periodic orbits—Quasi Halo orbits and Lissajous orbits. Quasi-halo orbits are quasi-periodic orbits around the halo orbits, and Lissajous orbits are quasi-periodic orbits around the collinear libration points. Shown in Figure 2 are one quasi-halo orbit (left) and one Lissajous orbit around the point L2. The case of the point L1 is similar.

However, the case of L3 is different. The out-of-plane amplitude of the halo orbit is very small, so do the quasi-halo orbits. The halo orbits and quasi-halo orbits are very close to the Lyapunov planar orbits. In practical applications, they can be seen as the Lyapunov planar orbits. Since the in-plane and

out-of-plane frequencies are very close, the Lissajous orbits are like periodic orbits. Shown in Figure 3 are one Lyapunov planar orbit and one Lissajous orbit around the point L3.
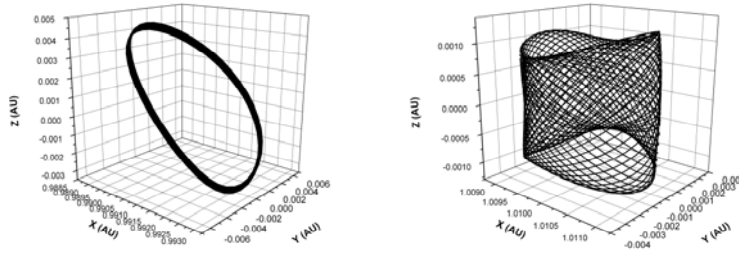

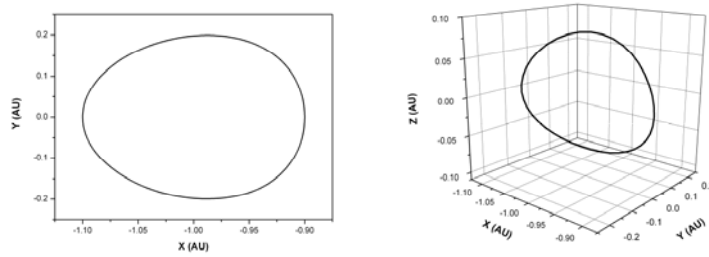Figure 2. Example halo orbit and Lissajous orbit around the point L2

Figure 3. Example Lyapunov planar orbit and Lissajous orbit around the point L3

The orbits around the collinear libration points can be used as the nominal orbits for the spacecrafts. There are invariant manifolds associated with them. In the following, for brevity, when we refer to the invariant manifolds of the collinear libration point Li, we mean the invariant manifolds of the periodic or quasi-periodic orbits around these points. For the points L1 and L2, their stable invariant manifolds can approach the earth. Shown in Figure 4 are the invariant manifolds of the points L1 and L2. The right figure is the local magnification of the left figure. If the stable invariant manifolds and the parking orbit near the earth can intersect, one impulse at the intersection point can send the spacecraft to the stable manifolds and wait for the spacecraft to evolve to the nominal orbits automatically [4]. This kind of transfer can be called direct transfer. Even if they can not intersect, a linking arc can be used to connect the parking orbit with the stable manifolds. However, the invariant manifolds of the point L3 can not approach the earth, as shown in Figure 5. Moreover, it spends thousands of years to leave the proximity of the point L3. So a direct transfer or a linking arc is not acceptable in transferring a spacecraft to the point L3. A possible way to fulfill the transfer is to send the spacecraft to the proximity of the point L3 first, and then change the speed to insert it into the nominal orbits. The unstable invariant manifolds of the point L2 pass through the proximity of the point L3 (as shown in the left figure of Figure 4), so the spacecraft can be sent to the point L3 via these manifolds. In the following, we pick L2 to state our strategies. The case of L1 is similar.
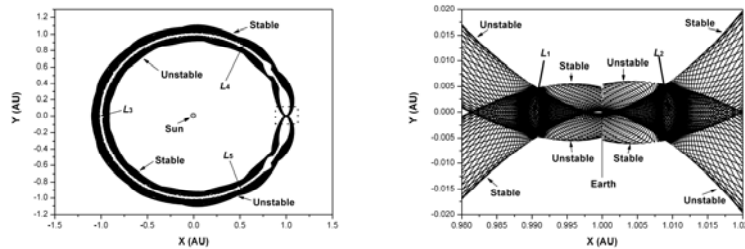

Figure 4. The invariant manifolds of the collinear libration points L1 and L2
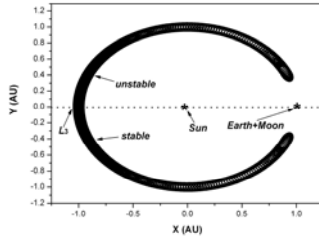
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Figure 5. The invariant manifolds of the collinear libration point L3

During the lifetime of the spacecraft, the communication between it and the earth may be shielded by the sun. A method to solve this problem is to put a relay station around the equilateral equilibrium point L_4 or L_5. Since the equilateral points are stable, the orbit control of the relay station is not energy-consuming. Fortunately, the unstable manifold of the point L_2 goes through the proximity of points L_5 (as shown in Figure 4), so we can transfer the relay station along with the spacecraft. When the carrier passes through the point L_5, the relay station is ejected and the remaining part goes on to the nominal orbit.

Figure 4 shows that the stable invariant manifold of the point L2 also passes through the proximity of the nominal orbits. With proper maneuvers, the spacecraft can depart from its nominal orbit and go back to the earth via the stable manifold of the point L_2. This fulfills a round trip to the earth. The orbit back to the earth can be seen as a mirror image of the transfer orbit to the point L3 due to the symmetry of the circular restricted problem [6], so we did not study it in the paper.

2.2 Dynamics of the equilateral libration points

The motion around the equilateral libration points is stable. The two eigenvalues of the motion in the $x-y$ plane are both imaginary numbers. According to Lyapunov's theorem, there are two families of periodic orbits in the $x-y$ plane---the long period family and the short period family. For the long period family, only the members very close to the equilateral libration points are stable, but all the members of the short period family are stable [8]. So we prefer the short periodic orbits as nominal orbits for the relay station. Shown in Figure 6 is an example short orbit. The dashed line is the line connecting the sun and the point L5. We use the distance λ between the point Q and the point L5 as the parameter of the two families, as Rabe did in his work [9]. For μ and λ small, the speed at point Q is approximately perpendicular to the dashed line.
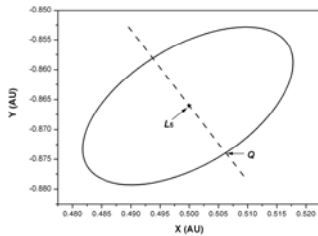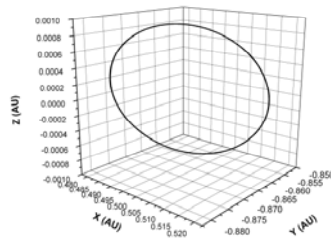

Figure 6. The planar short periodic orbit

Figure 7. The spatial quasi-halo orbit

Since the out-of-plane motion is stable libration, the motion is stable in 3D space if the motion projected onto the $x-y$ plane is stable. And since the out-of-plane frequency and the short frequency are very close, the motion in 3D space is like a periodic orbit if the motion in the $x-y$ plane satisfies

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the short periodic conditions. Shown in Figure 7 is an example orbit of this kind.

The periodic orbits in the $x-y$ plane and the quasi-periodic orbit in 3D space can be used as nominal orbits of the relay station.

2.3 Nominal orbits under the real solar system

When we refer to the real solar system, we mean the positions of the sun and the major planets are given by the numerical ephemeris. The ephemeris used in this paper is DE405. Although the motion around the point L5 is stable and instability of the motion around the point L3 is very weak, the periodic or quasi-periodic orbits obtained under the circular restricted three-body problem will deviate slowly under perturbations in the real solar system. However quasi-periodic orbits lasting for some time under the real solar system can be got using numerical algorithms [2]. Shown in Figure 8 are quasi-periodic orbits around the point L3 for 10 years. The initial MJD=54594. The initial positions in the synodic frame are the same as the Lyapunov planar orbit in Figure 3. The speed correction with respect to the speed under the circular restricted three-body problem is of the order 143m/s.
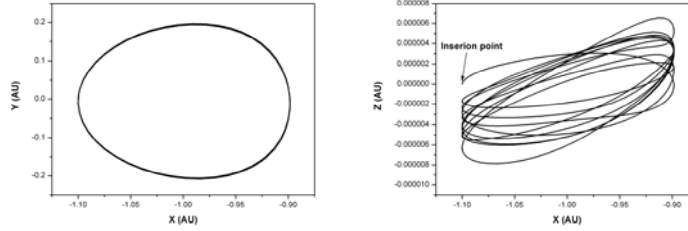

Figure 8. The quasi-periodic orbit around L3 constructed under the real solar system, 10 years

3. Methodology and Results

The simpler case planar circular restricted three-body problem was firstly studied. For the planar case, we choose Lyapunov planar orbits around L3 and short periodic orbits around L5 as nominal orbits. We denote the nominal orbit around L3 as nominal orbit A and the nominal orbit around L5 as nominal orbit B. The intersection point of the unstable manifolds with the x axis is chosen as the insertion point to nominal orbit A, and the intersection point of the unstable manifolds with the line connecting the sun and the L5 point (we denote this line as the line S-L5 in the paper) is chosen as insertion point to nominal orbit B. The initial conditions of the nominal orbits at these intersection points can be calculated, and the speed of the unstable invariant manifold at these intersection points can also be got by integration of the manifold. The speed corrections can be got from the differences of the two.

Denote the intersection point of the Lyapunov planar orbit with the x axis as point P. Since the Lyapunov orbit is symmetric with the x axis, the speed component of the point P along the x axis is always zero, but the speed component around the y axis direction is larger than zero, as shown in the right figure of Figure 9. Due to the mild nonlinear instability, the relation between $\dot{y}$ and x is nearly linear. The intersection of the unstable manifold with the x axis is isomorphic to circles [10], as shown in the left figure of Figure 9. The curve of $\dot{y}$ versus x is also shown in the right figure of Figure 9. Obviously, there is a minimal speed difference between the unstable manifolds and the nominal orbits along the y direction. At this particular insertion point, $\dot{x}=0$. The minimal speed correction intersection points are around the points with $\dot{x}=0$ in the left figure.

Similar curves can be got around the point L5. Different from nominal orbits A, the speed component of nominal orbit B along the line S-L5 is not zero (Although it is very small). It increases with the increasing amplitude of the nominal orbits, as shown in the left figure of Figure 10. The right figure is the curve of v_t versus r, where r is the distance from the sun and v_t is the speed vertical to the vector $\vec{r}$. Similar to the curve of the Lyapunov family, the curve shows some linearity. It's easy to show the minimal speed correction intersection points are around the intersection points of the two curves in the left figure.

Figure 9. The intersection curves of the invariant manifolds with the x axis and the initial conditions of the Lyapunov family around L3
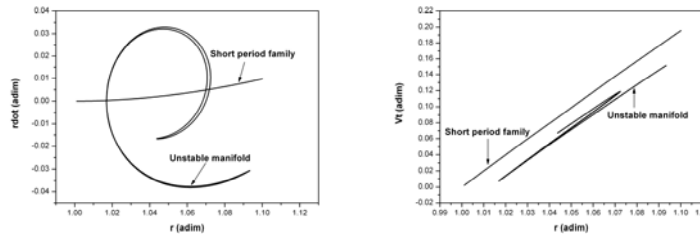

Figure 10. The intersection curves of the invariant manifolds with the line S-L5 and the initial conditions of the short period family around L5

Denote the speed corrections at nominal orbits B and A as Δv_2 and Δv_3, we choose the minimal $\Delta v = (\Delta v_2 + \Delta v_3)$ as the transfer orbit. Obviously, the speed corrections depend on the intersection curves of the unstable invariant manifolds. For different periodic orbits around the point L2, the energy of the unstable invariant manifolds is different, so is the intersection curve. The minimal energy-consuming transfer orbit and the time for transfer are different. Our computation shows that the minimal energy is smaller for larger periodic orbit around L2. But the time for the minimal energy-consuming transfer is longer. Shown in Figure 11 are the curves of Δv versus transfer time T for two periodic orbits around L2. The solid line is for $C = 3.000869$, and the dotted line is for $C = 3.000714$. The shorter the transfer time is, the larger energy is needed. The larger the energy of the manifold is, the minimal time for the transfer is shorter, and the maximal time for the transfer is longer.
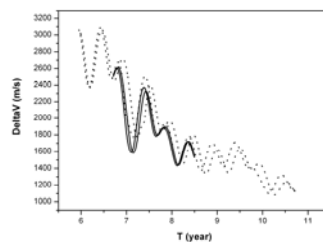

Figure 11.The curves of Δv versus T for different amplitude periodic orbits around L2

Shown in Figure 12 are the minimal energy-consuming transfer trajectories for invariant manifolds of energy $C = 3.000811$. The right figure is the local magnification of the left figure. The spacecraft is transferred from a 200km LEO around the earth (in the following, all the transfer orbits are from a 200km LEO). The speed corrections are $\Delta v_1 = 3212.35$m/s, $\Delta v_2 = 661.11$m/s, $\Delta v_3 = 672.40$m/s. The total energy consumed is 4545.86m/s. Transfer to the Lyapunov planar orbit around L2 needs time 0.7109years, and transfer from the Lyapunov planar orbit around L2 to the nominal orbit around L3 needs time 8.6877years. Very little energy is needed to insert the spacecraft to and depart from the periodic orbit around L2.
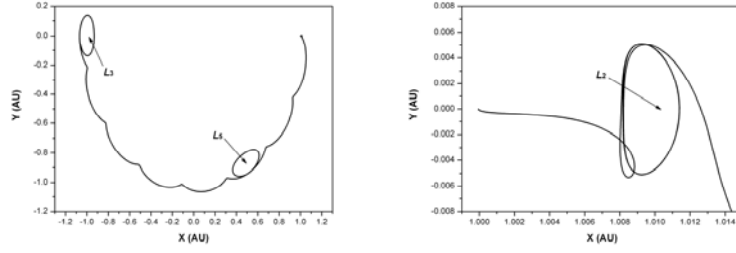

Figure 12. The transfer trajectory to L3 via the L2 gate (planar case)

Transfer through the L1 gate is similar to the case of L2. Shown in Figure 13 is the transfer trajectory through the L1 gate. $\Delta v_1 = 3212.36$m/s, $\Delta v_2 = 600.40$m/s, $\Delta v_3 = 577.07$m/s. The total energy consumed is 4389.83m/s. Transfer to the L1 periodic orbit needs time 0.7000 years, and transfer from the periodic orbit to the nominal orbit around L3 needs time 9.0847 years.
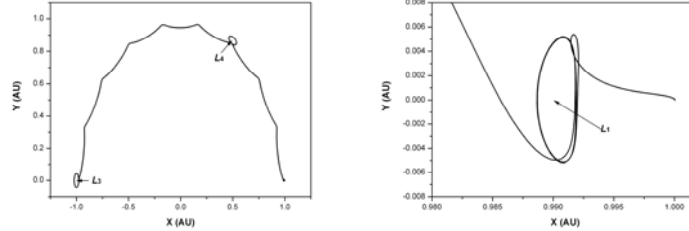

Figure 13. The transfer trajectory to L3 via the L1 gate (planar case)

For the spatial case, the intersection point of the manifold with the $x - z$ lane is chosen as insertion point to nominal orbit A, and the intersection point of the manifold with the plane which is vertical to $x - y$ plane and passes through the sun and the point L5 (we call this plane S-L5 plane) is chosen as insertion point to nominal orbit B. We change the speed along the x and y axes according to the relation of the Lyapunov planar family and the short period family, but leave the component along the z axis unchanged. Since the motion along the z axis is stable libration, the spacecraft can be kept around the L3 and L5 points without changing $\dot{z}$. Shown in Figure 14 is the transfer trajectory in the spatial case. The trajectory is similar to the planar case, so we just give the trajectory in the $x - z$ plane. Shown in Figure 15 are the nominal orbits around L3 (left) and L5. $\Delta v_1 = 3213.00$m/s, $\Delta v_2 = 666.69$m/s, $\Delta v_3 = 674.90$m/s, the total energy needed is 4554.59m/s. Transfer to the halo orbit around L2 needs time 0.7186 years, and transfer from the halo orbit around L2 to the nominal orbit around L3 needs time 8.7190 years.
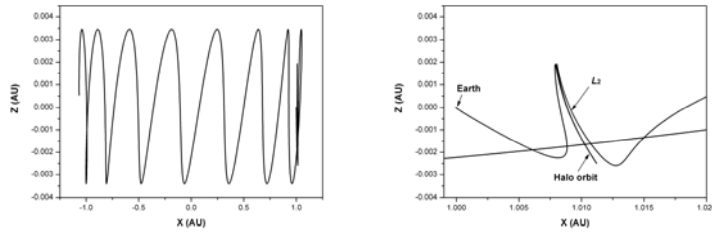
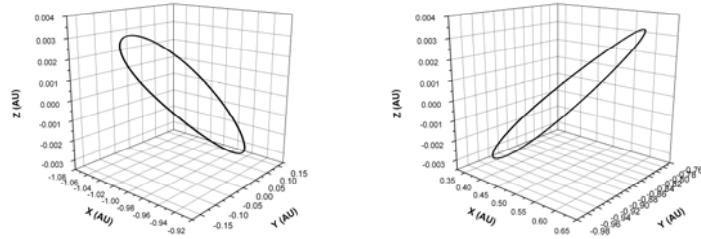Figure 14. The transfer trajectory to L3 via the L2 gate (spatial case)

Figure 15. The nominal orbits around L3 (left) and around L5

These figures show that part of the transfer time was spent on the periodic loops around L1 or L2. A possible way to avoid this is to choose transfer obit "inside" the tubes of the manifolds [3]. All the transfer orbits from the earth to the collinear libration point L3 via the L2 gate will pass through the $\eta - \zeta$ plane as shown in the left figure of Figure 16. Given C around C_2 and a point P in the $\eta - \zeta$ plane, the speed v can be got from Equation 2. Given θ and ψ, the initial conditions of the spacecraft can be got. Change the four variables η, ζ and θ, ψ, we can get different transfer orbits to the point L3. With a small maneuver at point P and integrate backwards, we can get a transfer orbit from the point P back to the earth. Shown in Figure 17 is one example transfer trajectory, where $C = 1.0444 \times 10^{-4}$. We just give the region around the point L2. $\Delta v_1 = 3212.63$m/s, $\Delta v_2 = 10.35$m/s, $\Delta v_3 = 698.14$m/s, $\Delta v_4 = 679.88$m/s. The total energy needed is 4601.00m/s. Transfer to point P needs time 0.4169 years, and transfer from point P to nominal orbit around L3 needs time 8.3030 years. Compared with the transfer orbit in Figure 12, less time is spent around the point L2.
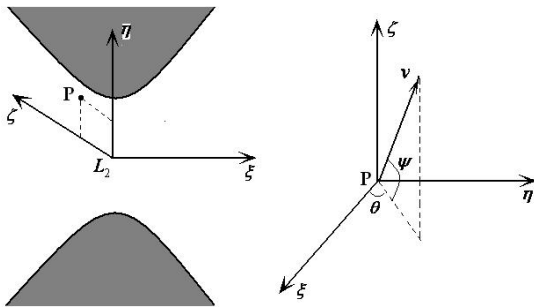

Figure 16. Illustration of finding transfer trajectory "inside" the unstable invariant manifolds
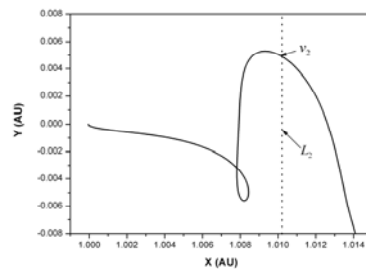
Figure 17. A transfer trajectory with four maneuvers

4. Conclusion

Concerning the use of the collinear libration point L3, the dynamics of the libration points were

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briefly studied. A way to send the spacecraft to the proximity of the points L3 and L5 was proposed. Some numerical simulation results were given. Although the works were done under the circular restricted three-body problem, the results obtained are useful when a transfer orbit under the real solar system is designed.

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